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Distance of attractors of evolutionary equations

(Distancia de atractores de ecuaciones de evolución)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Esperanza Santamaría Martín

Director

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Memoria para optar al título de Doctor

Distance of Attractors of Evolutionary Equations

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Director:
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Presentada por:
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Madrid, Octubre 2013

A mis padres
A Diego

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0.1. Introducción

Para hablar sobre los comienzos del estudio de los sistemas dinámicos tenemos que remontarnos a finales del siglo diecinueve. Henri Poincare, junto con Alexandre Mikhailovich Lyapunov y George David Birkhoff se convirtieron en los cofundadores de una nueva e importante área de investigación para aquel momento: Los Sistemas Dinámicos.

El concepto de sistema dinámico que actualmente conocemos fue desarrollado por Birkhoff a principios del siglo veinte. En aquella época los sistemas dinámicos que se analizaban eran de dimensión finita asociados a ecuaciones diferenciales ordinarias. Algunos de esos primeros estudios fueron sobre ciertos aspectos de estabilidad del sistema, como por ejemplo la teoría de un funcional de energía, la función de Lyapunov, que nos permite estudiar la estabilidad de ciertos sistemas, la teoría de conjuntos minimales, etc. También hubo otras aportaciones importantes como los métodos de construcción de variedades invariantes para problemas no lineales: El método de Hadamard (1901) y el método de Lyapunov (1928).

Con todas estas contribuciones, alrededor de 1930, quedó bien estructurada una teoría básica de sistemas dinámicos. A partir de ahí, aparecieron los primeros trabajos sobre la dinámica a largo tiempo de sistemas dinámicos asociados a ecuaciones diferenciales, incluyendo teoría de perturbación para variedades invariantes, dicotomías exponenciales, estructuras hiperbólicas, sistemas dinámicos Morse-Smale, entre otros.

La observación simple pero importante de considerar una ecuación de evolución como una ecuación diferencial ordinaria en un espacio de dimensión infnita llevó a algunos investigadores del momento a estudiar la dinámica de soluciones de ecuaciones en derivadas parciales. Así, muchos científicos han aplicado con éxito las técnicas de la teoría finito dimensional a la teoría infinito dimensional de sistemas generados por ecuaciones en derivadas parciales. Con esto, pronto surgió el estudio del comportamiento a largo tiempo de ecuaciones de evolución infinito dimensionales generadas por ecuaciones en derivadas parciales. Esta fusión del estudio finito dimensional e infinito dimensional se convirtió hacia 1970 en una única área: Dinámica

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de ecuaciones de evolución. Algunos de los matemáticos que contribuyeron en esta fusión fueron Jack K. Hale, [25], R. Teman, [54], A. V. Babin y M. I. Vishik, [13] y O. A. Ladyzhenskaya, [36].

Una clase muy importante de sistemas dinámicos de dimensión infinita son los llamados disipativos, cuyo nombre se debe a que estos sistemas poseen cierto mecanismo que no permite estados con energía muy alta y se caracterizan por la existencia de un conjunto acotado tal que la órbita de cada estado inicial después de un tiempo suficientemente grande entra en este conjunto acotado. Debido a esta propiedad disipativa y con cierta compacidad, se puede probar que la dinámica asintótica de un sistema dinámico disipativo se concentra en un conjunto compacto e invariante llamado atractor. Este conjunto contiene toda la información de la dinámica asintótica del sistema. Es por esto que la existencia y análisis de este conjunto atractor han sido muy estudiados por muchos matemáticos. Durante décadas, se podía encontrar en la literatura trabajos sobre propiedades importantes de este objeto. De hecho, hay muchos autores que han trabajado concienzudamente por entender bien su estructura, la dinámica que contiene así como su dimensión fractal, entre otras propiedades.

Una propiedad clave del atractor es su robustez respecto a perturbaciones. Por ello, se pueden encontrar muchos trabajos que analizan el comportamiento del atractor bajo perturbaciones del sistema. Estas perturbaciones pueden ser regulares o singulares, pueden afectar al dominio. El análisis del comportamiento del atractor bajo perturbaciones se puede dividir en dos tipos.

i) Probar la continuidad de los tractores bajo perturbaciones, es decir, obtener que los atractores de los problemas perturbados están, en cierta métrica, "cerca" del atractor no perturbado. Una clase de sistemas para los cuales se han obtenido buenos resultados en este aspecto son los sistemas gradientes, donde el atractor tiene una determinada estructura (el atractor sólo se compone de puntos de equilibrio y conexiones entre ellos).

ii) Estimar, en cierta métrica, la distancia de estos atractores una vez que sabemos que se comportan de forma continua.

Esta tesis se centra en este segundo tipo de análisis.

Respecto al primero, existen muchas aportaciones en la literatura como por ejemplo la tesis doctoral de G. Cooperman en 1978, [20], donde el autor obtuvo la semicontinuidad superior de atractores locales bajo ciertas hipótesis. Este trabajo fue el punto de partida de otros muchos como el de J. Hale, [24], que estudió la semicontinuidad superior de atractores para sistemas gradientes. Más tarde Hale, Magalhães y Oliva, [27], probaron la continuidad de los atractores, semicontinuidad superior e inferior, cuando el sistema no perturbado es Morse-Smale, es decir, cuando el sistema tiene un número de equilibrios finito, todos ellos hiperbólicos, cuyas variedades estables e inestables intersecan de manera transversal. Años después, Hale y Raugel en

[30], presentaron un resultado de semicontinuidad inferior cuando el sistema límite es gradiente con todos los puntos de equilibrio hiperbólicos. Este trabajo es un referente en el estudio de la semicontinuidad inferior de los atractores.

Con todos estos resultados de continuidad de atractores, a finales de los años 80, surgió de forma natural el problema de estimar la tasa de convergencia de estos atractores. El primer resultado al respecto destacable es debido a Babin y Vishik, [13]. Ellos prueban que si los atractores atraen exponencialmente y la distancia de los semigrupos es de orden ε, entonces la distancia de los atractores es de orden ε^q con $q < 1$. Otra aportación relevante en el estudio de la tasa de convergencia de los atractores es el resultado de Hale y Raugel, [30], donde obtienen un resultado análogo al de Babin y Vishik. La diferencia es que en lugar de considerar atractores exponencialmente atrayentes, se centran en perturbaciones de sistemas gradientes con todos los puntos de equilibrio hiperbólicos y usan la atracción exponencial de las variedades inestables locales.

Es importante destacar cómo en ambos resultados la tasa de convergencia de los atractores no parece ser óptima. De hecho, en muchos casos de sistemas gradientes, por ejemplo [5], la distancia entre los puntos de equilibrio del problema perturbado y el problema no perturbado (problema límite) es del mismo orden que la distancia de los semigrupos y lo mismo ocurre con la distancia de las variedades inestables locales. Sin embrago, en el proceso de poner todos estos ingredientes juntos hay cierta pérdida que hace que la tasa de convergencia de los atractores obtenida mediante estas técnicas sea peor que la distancia de los semigrupos.

0.2. Objetivo

El objetivo y principal motivación de esta tesis es estudiar si, al menos para algún tipo de sistema, podemos obtener la distancia de atractores del orden de la distancia de los semigrupos. En este trabajo probamos que podemos obtener una tasa de convergencia de los atractores de ese orden para sistemas gradientes Morse-Smale.

0.3. Contenido

En este trabajo obtenemos que, si tenemos un sistema finito dimensional generado por una aplicación Morse-Smale gradiente la cual tiene un atractor y perturbamos este sistema de forma que el sistema perturbado tiene también un atractor, enonces la tasa de convergencia de los atractores es de orden de la distancia de las aplicaciones a tiempo uno correspondientes. Además, generalizamos este resultado mediante el siguiente: Si tenemos un sistema dinámico finito dimensional generado por una aplicación Morse-Smale que tiene un atractor y consideramos una perturbación de él no autónoma de forma que, el sistema dinámico perturbado no autónomo tiene un atractor pullback, entonces la tasa de convergencia del atractor y del atractor pullback es de orden de la tasa de convergencia de las aplicaciones a tiempo uno correspondientes. Estos resultados son obtenidos en el caso finito dimensional, es decir, básicamente para aplicaciones a tiempo uno en $\mathbb{R}^m$ y la prueba consiste en usar las propiedades de la teoría de Shadowing que tienen las aplicaciones Morse-Smale. Existen varios trabajos en la literatura al respecto, como [41, 42], que muestran que una aplicación Morse-Smale tiene la propiedad de Lipschitz Shadowing. Nosotros además veremos que también tienen la propiedad de lo que hemos llamado "Nonautonomous inverse shadowing". Esta última propiedad nos va a permitir conseguir resultados en esta dirección para perturbaciones no autónomas. Un aspecto que aparentemente no se ha desarrollado mucho es la conexión entre estas propiedades de Shadowing y la distancia de atractores que, por otro lado, surge de forma natural de la definición de shadowing. Nosotros utilizamos esta relación para obtener buenas estimaciones de la distancia de atractores. Todo esto está descrito con detalle en el Capítulo 1.

Como el resultado obtenido de distancia de atractores utilizando propiedades de shadowing es finito dimensional y nosotros queremos aplicar esta técnica a sistemas infinito dimensionales, por ejemplo ecuaciones de evolución del tipo $u_t + Au = F(u)$ en un espacio de Hilbert X, necesitamos una herramienta que reduzca este sistema a uno de dimensión finita. Una herramienta apropiada para esto son la Variedades Inerciales, es decir, variedades diferenciales, positivamente invariantes bajo el flujo, de dimensión finita que atraen exponencialmente. En particular, si el sistema tiene un atractor, entonces la variedad inercial lo contiene. Estas variedades son construidas como un grafo de una función regular $\Phi : V \to W$ con dominio un subespacio vectorial de dimensión finita V y rango el complemento ortogonal de este espacio, W el cual es un subespacio cerrado de dimensión infinita. De hecho, el espacio V suele estar generado por las m autofunciones asociadas a los primeros m autovalores del operador A. La construcción de la variedad no es trivial, se necesita que los autovalores de A satisfagan una cierta condiciòn que se llama "gap condition". Este es el mayor inconveniente de esta teoría pues no es frecuente que esta condición se satisfaga. De todas formas, para algunas ecuaciones de evolución en un dominio de una dimensión este "gap" en los autovalores existe lo que abre la posibilidad de usar la técnica mencionada. Nosotros queremos estudiar el comportamiento de sistemas bajo perturbaciones y obtener buenas tasas de convergencia de sus atractores, por lo tanto, necesitaremos estudiar el comportamiento de estas variedades inerciales bajo perturbaciones del sistema. Estudiamos este aspecto en el Capítulo 2, donde consideramos la familia de sistemas del tipo $u_t + A_\varepsilon u = F_\varepsilon(u)$ en el espacio de fase X_ε y el sistema $u_t + A_0 u = F_0(u)$ en el espacio de fase X_0 y obtenemos buena tasa de la distancia de las variedades inerciales tanto en la topología C^0 como en la topología $C^{1,\theta}$. Como los espacios X_ε y X_0 son diferentes, necesitamos una forma de pasar la información de un espacio a otro mediante ciertos operadores $E : X_0 \to X_\varepsilon$ y $M : X_\varepsilon \to X_0$. Veremos que si somos capaces de estimar la distancia de los operadores resolventes

$$\|A_\varepsilon^{-1} - E \circ A_0^{-1} M\| \leq \tau(\varepsilon)$$

y las no linealidades

$$\|F_\varepsilon \circ E - E \circ F_0\| \leq \rho(\varepsilon),$$

entonces las variedades inerciales, las cuales vienen dadas por el grafo de funciones Φ_ε y Φ_0, satisfacen

$$\|\Phi_\varepsilon - E\Phi_0\| \leq C(\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)).$$

Referimos a la introducción del Capítulo 2 y especialmente a las Secciones 2.1.1 y 2.2.1 para una descripción más detallada de estos resultados.

El factor $\log(\tau(\varepsilon))$ que aparece en la tasa (ver Teorema 2.1.4 y Teorema 2.2.2) aparentemente parece que aparece por causas técnicas, sin embargo, con los métodos que hemos usado no hemos sido capaces de eliminarlo.

Los dos primeros capítulos son independiente uno de otro y se puede leer en cualquier orden. Estos capítulos introducen dos herramientas que usaremos para alcanzar el principal objetivo de la tesis.

Finalmente, en el Capítulo 3, aplicaremos las técnicas desarrolladas en los capítulos anteriores para analizar un problema de distancia de atractores correspondientes a un sistema dinámico generado por una ecuación de reacción-difusión

$$\begin{cases} u_t - \Delta u + \alpha u = f(u) & \text{en} \quad Q_\varepsilon, \\ \frac{\partial u}{\partial \nu_\varepsilon} = 0 & \text{en} \quad \partial Q_\varepsilon, \end{cases}$$

en un dominio fino del tipo

$$Q_\varepsilon = \{(x, \varepsilon \mathbf{y}) \in \mathbb{R}^d : (x, \mathbf{y}) \in Q\}, \qquad \varepsilon \in (0,1),$$

donde Q es un conjunto dado en $\mathbb{R}^d$, (por ejemplo $Q = \{(x, \mathbf{y}) : |\mathbf{y}| \leq r(x)\}$ con $r : (0,1) \to \mathbb{R}^+$; este dominio representa un canal fino con secciones transversales circulares de radio $r(x)$). No nos restringiremos a este tipo de canales finos, de hecho consideraremos dominios finos con secciones transversales no necesariamente circulares, pero sí hay que tener en cuenta que esta es una clase de dominios finos muy importante. Observese que el factor ε en las coordenadas $\mathbf{y}$ hace que el dominio Q_ε se aproxime al segmento $[0,1]$ cuando $\varepsilon \to 0$.

El problema límite es el siguiente sistema de dimensión uno

$$\begin{cases} u_t - \frac{1}{g}(g u_x)_x + \alpha u = f(u) & \text{en} \quad (0,1), \\ u_x(0) = u_x(1) = 0. \end{cases} \tag{0.3.1}$$

ver [31, 46] entre otros.

Se sabe, [33], que la aplicación a tiempo uno del sistema límite (0.4.6) es una aplicación Morse-Smale gradiente. Además el operador lineal elíptico asociado al problema límite es de tipo Sturm-Liouville, lo cual nos garantiza que tenemos un espacio apropiado en el espectro. Estas dos propiedades nos permiten aplicar las técnicas de variedades inerciales para reducir nuetsro sistema a uno de dimensión finita (usando ls resultados del Capítulo 2) y aplicar las técnicas de shadowing para estimar la distancia de los atractores (Capítulo 1). Con estas dos técnicas somos capaces de obtener la siguiente estimación para la distancia de atractores,

$$\mathrm{dist}_{H^1(Q_\varepsilon)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq C\varepsilon^{\frac{d+1}{2}} |\log(\varepsilon)|$$

la cual mejora la obtenida en [31]. Además, esta estimación parece óptima salvo por el factor $|\log(\varepsilon)|$.

Al final, hemos incluido dos apéndices. En el Apéndice A describimos una colección de resultados conocidos que relacionan, como hemos mencionado antes, la estructura Morse-Smale y la propiedad de shadowing llamada Lipschitz shadowing.

Y en el apéndice B presentamos la prueba de la relación entre la estructura Morse-Smale y y las propiedades de shadowing no autónomas que, aunque no se encuentra en la literatura, sigue los argumentos de [35].

0.4. Conclusiones

Los resultados presentados en esta tesis nos permiten obtener una buena estimación de la tasa de convergencia de los atractores asociados a un sistema finito dimensional generado por una aplicación Morse-Smale y una perturbación autónoma del mismo asi como una perturbación no autónoma. Además, en el caso en que tengamos un sistema que posee una variedad inercial y lo perturbamos de forma que el perturbado también tenga variedades inerciales, somos capaces de obtener una buena estimación de la distancia de esas variedades. Con esto, podemos dar una buena estimación para la distancia de atractores de un sistema Morse-Smale infinito dimensional y una perturbación de él, reduciendo los sistemas a sistemas finito dimensionales mediante variedades inerciales. Estas estimaciones mejoran las que se conocían actualmente en la literatura.

Introduction

To speak about the beginning of the study of dynamical systems, we would have to go back to the end of the nineteenth century. Henri Poincare, along with Alexander Mikhailovich Lyapunov and George David Birkhoff became the co-founders of a new an important area at that time: Dynamical Systems.

The concept of dynamical system we use now was developed by Birkhoff at the beginning of the twentieth century. The main setting at that time was in the framework of finite dimensional dynamical systems coming from ordinary differential equations. Some of these works were about the stability, in certain sense, of the dynamical system, as for instance, the theory of generalized energy functional, Lyapunov function, which allows us to study the stability of some systems and the theory on minimal sets, nonwandering sets, etc. But, of course, in that beginning, there were advances due to others mathematicians, too. We note here the two methods for the constructions of invariant manifolds for nonlinear problems: the Hadamard method (1901) and the Lyapunov (1892)-Perron (1928) method.

With all of these contributions, around 1930 a basic theory of dynamical systems was well structured. Since then, much of the developments in the analysis of the longtime dynamics of systems of ordinary differential equations took place, including perturbation theory for invariant manifolds, exponential dichotomies, hyperbolic structures, Morse-Smale dynamical systems and so forth.

The simple, but very deep observation of considering an evolutionary equation as an ordinary differential equation in an infinite dimensional space, say a Banach space, led some researchers to study the dynamics of solutions of partial differential equations. Many scientists have successfully applied ideas and notions from the finite dimensional theory to the theory of infinite dimensional systems coming from partial differential equations. Early, the study of the longtime behavior of infinite dimensional evolutionary equations generated by partial differential equations arose. Hence, the merger of the study of finite dimensional and infinite dimensional dynamical systems can be dated around 1970, becoming in an unique area: the Dynamics of Evolutionary Equations. Some of the mathematicians that contribute in this merger were Jack K. Hale, [25], R. Teman, [54], A. V. Babin and M. I. Vishik, [13] and O. A. Ladyzhenskaya, [36].

A very important class of infinite dimensional dynamical systems are those called dissipative, which receive this name because roughly speaking they have certain mechanism which does not allow states with very high energy and are characterized by the existence of a bounded set, maybe large, so that the orbit of each initial state enters into this bounded set if we let time pass long enough. Because of this mechanisms and with some compactness it can be proved that, the asymptotic behavior of a dissipative dynamical system is concentrated in a compact invariant set called attractor. This set, carries all the information of the asymptotic dynamics of the system and the existence and analysis of it has attracted a lot of attention. For decades, the literature was full of works about some properties of this important object. In fact, there are many authors who have worked very hard to understand well its fine structure, the dynamics inside it and other relevant properties like its fractal dimension and so forth.

One of the key properties of the attractor is its robustness with respect to perturbation and we find quite a few results that analyze the behavior of the attractor with respect to perturbations of the system. These perturbations may be regular or singular, may affect the domain where the physical process takes places or even change drastically the type of the equation. The analysis and results on the behavior of the attractors under perturbations may be divided in two classes.

i) To show the continuity of the attractors under perturbations, that is, to obtain that the attractors of the perturbed problems are "near" in certain metric, to the attractor of the unperturbed one. A class for which good results on continuity has been obtained is the class of gradient systems, for which the attractor has some special structure (it consists only of equlibria and connections among them)

ii) To obtain estimates of the distance of these attractors in certain metrics, once we know they behave continuously.

This thesis is devoted to this second class of questions. We will show that for a more restrictive class of systems (Morse-Smale gradient systems) we will be able to obtain good estimates on the distance of these attractors. Moreover, we will be able to apply the results to a relevant perturbation problem, which is a reaction diffusion equation in a thin domain. We will be able to obtain rates of the distance of the attractors which improve previous results in the literature (see [31]) and moreover, the rates we will obtain will be basically optimal. Two important tools we will need to obtain our result are Shadowing techniques and Inertial Manifolds.

To be more specific with the framework of the thesis, we first present some definitions and results. The basic concept involved in the study of infinite dimensional dynamical systems is the concept of nonlinear semigroup. We start with this definition,

Definition 0.4.1. *(A. V. Babin and M. I. Vishik, [13]) Let V be a metric space. A one parameter family $\{T_0(t)\}$ of maps $T_0(t) : V \to V$, $t \geq 0$, is called a* $\mathbf{C^0}$-***semigroup*** *if*

(1) $T_0(0)$ *is the identity map on* V,

(2) $T_0(t+s) = T_0(t)T_0(s)$ *for all* $t, s \geq 0$*,*

(3) the function

$$[0,\infty) \times V \ni (t,x) \to T_0(t)x \in V$$

is continuous at each point $(t,x) \in [0,\infty) \times V$*.*

The notion of attractor is the following,

Definition 0.4.2. *(Jack K. Hale, [25]) A set* $\mathcal{A}_0 \subset V$*, with* V *a metric space, is called a* ***global attractor*** *for the semigroup* $\{T_0(t)\}$ *on* V *if*

(1) $\mathcal{A}_0$ *is nonempty, compact, and invariant with respect to* $\{T_0(t)\}$*;*

(2) $\mathcal{A}_0$ *attracts each bounded set of* V*.*

The framework of the continuity problem is the following. Suppose we have a family $\{T_\varepsilon(t) : \varepsilon \geq 0\}$ of semigroups on X, with X a Banach space. Also assume that each $T_\varepsilon(t)$ has a compact attractor $\mathcal{A}_\varepsilon$, $\varepsilon \geq 0$. We say $\mathcal{A}_\varepsilon$ is *upper-semicontinuous* at $\varepsilon = 0$ if,

$$d(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Moreover, we say $\mathcal{A}_\varepsilon$ is *lower-semicontinuous* at $\varepsilon = 0$ if,

$$d(\mathcal{A}_0, \mathcal{A}_\varepsilon) \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Where,

$$d(A,B) = \sup_{a\in A} d(a,B) \qquad \text{and} \qquad d(a,B) = \inf_{b\in B} \|a-b\|_X. \tag{0.4.1}$$

We say $\mathcal{A}_\varepsilon$ is *continuous* at $\varepsilon = 0$ if it is upper and lower semicontinuous at $\varepsilon = 0$, that is if

$$dist_H(\mathcal{A}_0, \mathcal{A}_\varepsilon) \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

where $dist_H$ is the symmetric Hausdorff distance of two sets, that is

$$dist_H(A,B) = \max\{d(A,B), d(B,A)\}. \tag{0.4.2}$$

We denote the symmetric Hausdorff distance by $dist_H$ or if we want to stress the space X we will denote it by $dist_X$.

The first result in this direction is due to G. Cooperman in 1978, [20], his Ph. D. Thesis. He got the upper-semicontinuity of local attractors provided the dependence of $T_\varepsilon(t)x$ on ε is continuous uniformly on bounded sets in (t,x). This work was the starting point of a large number of papers related to this problem. In 1985, Hale studied this problem for gradient systems in [24]. Few years later, Hale, Lin and Raugel in [26], obtained upper-semicontinuity of local attractors with a weaker dependence of $T_\varepsilon(t)$ on ε than in Cooperman work. In their result, the semigroups $T_\varepsilon(t)$ need only approximate T_0 in an appropriate sense, which is so general that the approximated semigroups can correspond to numerical schemes for evolutionary equations. Also, Hale and Raugel have studied the upper-semicontinuity of the attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$ for a hyperbolic equation which degenerates to a parabolic equation for $\varepsilon = 0$, see [29]. A more difficult question was the lower-semicontinuity of attractors. Additional conditions on the limit flow restricted to $\mathcal{A}_0$, are needed. Hale, Magalhães and Oliva, see [27], proved the attractors $\mathcal{A}_\varepsilon$ are continuous at $\varepsilon = 0$, upper and lower-semicontinuous, if the limit semigroup $T_0(t)$ is Morse-Smale. That is, if $T_0(t)$ has a finite number of equilibrium points, all of them hyperbolic with the stable and unstable manifolds transversal. This required property, implies more information than lower-semicontinuity. The hyperbolicity of the equilibria and the transversal intersection of stable and unstable manifolds imply the flow of the attractor is stable under smooth perturbations of the semigroup. Also, the hyperbolicity of all equilibrium points is a local property, which is verified analyzing the eigenvalues of some linear operator, whereas the transversality is a global property for which there is no a general procedure to verify it. At that time, intuition led them to think it should be possible to obtain continuity of attractor without transversality. Some years later, Hale and Raugel in [30], presented a class of semigroups $T_\varepsilon(t)$ for which one has the lower-semicontinuity property. That class of semigroups is that one whose limit at $\varepsilon = 0$ is a gradient system and with all equilibrium points hyperbolic. The property of gradient system is often easy to verify in applications. This paper, [30], has became in a reference work to study the lower-semiconinuity of attractors. The main result described in this work consists in considering the following hypoteses: the limit semigroup $T_0(t)$ is a gradient system with all equilibrium points hyperbolic, the equilibrium points of the perturbed system converge to the limit ones, the local unstable manifolds are lower semicontinuity and there exists an estimate for the distance of semigroups far away the neighborhoods of equilibrium points. With these hypotheses, the authors obtain lower semicontinuity of attractors.

This result is general enough to be applied to numerical approximations of parabolic equations or to singularly perturbed problems. The main idea of the proof is to “transfer” the lower-semiconinuity of the local unstable manifolds to the global unstable manifolds and so, to have lower-semicontinuity of attractors. This procedure is carried out taking into account the continuity property of the equilibrium points and of local unstable manifolds under perturbation, using a Morse decomposition for the limit attractor $\mathcal{A}_0$.

Babin and Vishik, [12], also obtained a continuity result of the attractors, but this time in the case where the semigroups depend continuously on the parameter ε.

There exist many works in the literature where these continuity of attractors results are applied to different situations. For instance, the case of a reaction-diffusion equation where the domain is perturbed has been analyzed in [4], where the continuity has been shown. One example can be found in the series of papers of the work by Arrieta, Carvalho and Lozada-Cruz, [6], [7] and [8]. In these three papers the authors study the behavior of asymptotic dynamics of a dissipative reaction-diffusion equation in a dumbbell domain, Ω_ε. That is, a domain Ω_ε, consisting in two disconnected domains, which they denote by Ω, joined by a thin channel, R_ε, that degenerates to a line segment as the parameter ε goes to 0. More precisely, they analyze the following parabolic equation

$$\begin{cases} u_t - \Delta u + u = f(u) & x \in \Omega_\varepsilon \quad t > 0, \\ \frac{\partial u}{\partial n} = 0 & x \in \partial\Omega_\varepsilon, \end{cases}$$

where $\Omega_\varepsilon \subset \mathbb{R}^N$, $N \geq 2$, with the limit problem

$$\begin{cases} w_t - \Delta w + w = f(w), & x \in \Omega \quad t > 0 \\ \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega \\ v_t - \frac{1}{g}(gv_x)_x + v = f(v), & x \in (0,1) \\ v(0) = w(P_0),\ v(1) = w(P_1) \end{cases}$$

Here w is a function defined in Ω, v defined in the linear segment $R_0 = \{x, 0, ..., 0 : 0 < x < 1\}$ and the points P_0 and P_1 are the points of the junction of the line R_0 with the open set Ω. The function g is related to the geometry of R_ε.

The authors show the attractors are continuous in certain metric. The singular character of the dumbbell perturbation makes the analysis of the problem extremely technical although the general arguments to obtain the continuity follow somehow the line of Hale and Raugel in [25] and [30].

With this great amount of continuity results of attractors, the problem of the rate of convergence of attractors arises naturally. The study of this problem goes back to the end of the eighties. First of all, we introduce the framework. Let $\{T_\varepsilon(t)\}_{\varepsilon\geq 0}$ be a family of semigroups associated to an evolution equation and a perturbation of it. We assume they have attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\geq 0}$. The first important result in this direction is due to A. V. Babin and M.I. Vishik. They obtained the following theorem,

Theorem 0.4.3. *(A.V. Babin and M.I. Vishik, [13]) Suppose there exists a bounded set B_0 such that*

$$\mathcal{A}_\varepsilon \subset B_0 \qquad \forall \varepsilon \geq 0, \tag{0.4.3}$$

and

$$d(T_\varepsilon(t)B_0, \mathcal{A}_\varepsilon) \leq Ce^{-\alpha t} \qquad \forall \varepsilon \geq 0, \tag{0.4.4}$$

where C and α do not depend on ε. Moreover, it is supposed that for any $\varepsilon \geq 0$ and for any $u_1, u_2 \in B_0$

$$\|T_\varepsilon(t)u_1 - T_0(t)u_2\| \leq C_1 e^{\beta t}\|u_1 - u_2\| + Ce^{\beta t}\varepsilon. \tag{0.4.5}$$

Then there exists such a constant C_2 that

$$dist_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq C_2\varepsilon^q, \qquad q = \frac{\alpha}{\alpha+\beta}.$$

Recall that $d(\cdot,\cdot)$ and $dist_H(\cdot,\cdot)$ are defined in (0.4.1) and (0.4.2), respectively.

The theorem above says that if any attractor $\mathcal{A}_\varepsilon$ attracts exponentially and we have the estimate (0.4.5) for the semigroups distance, then they obtain an estimate for the attractors distance of order q with $q < 1$. The idea of the proof is to look for an optimal time t_0 so that the combination of the two estimates (0.4.4) and (0.4.5) be optimal in time.

Other important contribution to the study of the rate of convergence of attractors is the result of J. K. Hale and G. Raugel, where instead of requiring the exponential attractivity of the attractors, they focus on perturbations of gradient systems with all equilibria hyperbolic and use the exponential attractivity of the local unstable manifolds.

The result of Hale and Raugel is the following

Theorem 0.4.4. *(J. K. Hale and G. Raugel, [30]) Let X be a Banach space. We assume,*

(1) $T_0(t)$, $t \geq 0$, is a C^1 gradient system which is asymptotically smooth,

(2) the set of equilibrium points of $T_0(t)$, $E_0 = \{u^*_{1,0}, ..., u^*_{N,0}\}$, is bounded and finite with each equilibrium point, $u^*_{j,0}$, hyperbolic,

(3) for $\varepsilon \neq 0$, $T_\varepsilon(t)$ is a C^1-semigroup and admits a local attractor $\mathcal{A}_\varepsilon$ attracting U_0, where U_0 is a fixed open neighborhood of $\mathcal{A}_0$ in X,

(4) let E_ε be the set of equilibrium points of $T_\varepsilon(t)$; there exists an open neighborhood W_0 of E_0 in X, such that $W_0 \cap E_\varepsilon = \{u^*_{1,\varepsilon}, ..., u^*_{N,\varepsilon}\}$ contains the same number of equilibrium points than E_0, all of them hyperbolic; moreover, $u^*_{j,\varepsilon} \to u^*_{j,0}$ in X as $\varepsilon \to 0$, and

$$\mathcal{A}_\varepsilon = \bigcup_{1 \leq j \leq N} W^u_\varepsilon(u^*_{j,\varepsilon}),$$

(5) there exist two positive constants C_0 and β such that, for any u_1, u_2 belonging to $\cup_{\varepsilon \geq 0} \mathcal{A}_\varepsilon$,

$$\|T_0(t)u_1 - T_0(t)u_2\|_X \leq C_0 e^{\beta t} \|u_1 - u_2\|_X,$$

(6) there is a positive real number p such that

$$dist_H(W^u_{loc,0}(u^*_{j,0}), W^u_{loc,\varepsilon}(u^*_{j,\varepsilon})) \leq C_0 \varepsilon^p,$$

for $1 \leq j \leq N$, with $W^u_{loc,\varepsilon}(u^*_{j,\varepsilon})$ the local unstable manifold of the equilibrium point $u^*_{j,\varepsilon}$, $\varepsilon \geq 0$,

(7) for any $t^*_0 > 0$, there exists a real number $C^*_0 \equiv C^*_0(t^*_0) > 0$ such that, for $u_\varepsilon \in \mathcal{A}_\varepsilon \cup U_0$,

$$\|T_0(t)u_\varepsilon - T_\varepsilon(t)u_\varepsilon\|_X \leq C^*_0 \varepsilon^p e^{\beta t} \qquad \text{for } t \geq t^*_0.$$

Then there are two positive constants C_1 and q, with $0 < q < p$, such that

$$dist_H(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq C_1 \varepsilon^q \qquad q = (\frac{\alpha}{\alpha+\beta})^{M-1} p,$$

where M is such that, $v^1 > v^2 > ... > v^M$ are the distinct points of the set $\{V(u^*_{1,0}), ..., V(u^*_{N,0})\}$ with V the Lyapunov function associated with the semigroup $T_0(t)$.

This result is proved by induction arguments. They classify all the equilibrium points in sets according to the levels energy of certain associated Lyapunov function. With this, they obtain the result taking the hyperbolic equilibrium point property, that is, the exponential attraction of the local unstable manifolds, and the fact that the associated Lyapunov function V is non-increasing on t, as the main tools.

In the mentioned result of J. K. Hale and G. Raugel,

$$dist_H(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq C_1 \varepsilon^q, \qquad \text{with} \quad q = (\frac{\alpha}{\alpha+\beta})^{M-1} p,$$

the term α, which appears in the exponent q, is determined by the infimum of the exponents related to the exponential attraction of the local unstable manifold of each hyperbolic equilibrium point.

There exist several applications of these results in the literature. For example, in [5] a family of parabolic equations is studied

$$\begin{cases} u^\varepsilon_t - div(a_\varepsilon(x)\nabla u^\varepsilon) = f(u^\varepsilon) & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \\ u^\varepsilon(0) = u^\varepsilon_0 \end{cases}$$

where the diffusion coefficients $a_\varepsilon : \Omega \to \mathbb{R}^m$ are smooth function which are uniformly lower bounded and upper bounded. It is assume that, for each $0 \leq \varepsilon \leq \varepsilon_0$, there exists an attractor $\mathcal{A}_\varepsilon$ and the rate of convergence of these attractors is analyze. Then, applying similar techniques than Babin and Vishik, Hale and Raugel, the author obtain the following rate of convergence

$$dist_H(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq C \|a_0 - a_\varepsilon\|^\gamma_{L^\infty(\Omega, \mathbb{R}^m)},$$

with $\gamma < 1$.

Note that, apparently in both results Theorem 0.4.3 and Theorem 0.4.4, the rate of convergence of attractors does not seem to be optimal. As a matter of fact, in many instances for gradient systems, see for instance [5], the distance between the equilibria of the perturbed problem and of the unperturbed problem is of the same order as the distance of the semigroups and the distance between the local unstable manifolds is also of the same order as the distance of the semigroup. Then, it seems that it is in the process of "gluing" all the information together that there is some loss and the rates obtained for the convergence of the attractors end up being much worse than the distance of the semigroups.

One of the main motivations for this Thesis is to decide whether, at least for some kind of systems, the distance of the attractors can be obtained of the same order as of the distance of the nonlinear semigroups. We will see that for Morse-Smale gradient systems, this can be obtained.

More specifically, what we prove in this work is that, if we have a finite dynamical system generated by a Morse-Smale gradient map which has an attractor and we perturb this system in such way so that the perturbed system has also an attractor, then the rate of convergence of the attractors is of order the distance of the corresponding time one maps. Moreover, we generalize this result with the following one: If we have a finite dynamical system generated by a Morse-Smale map which has an attractor, and we consider a non-autonomous perturbation of it, such that this non-autonomous perturbed dynamical system has a pullback attractor, then the rate of convergence of the attractor and pullback attractor is of order the rate of convergence of the corresponding time one maps. These results are obtained in a finite dimensional framework, that is, basically for time one maps in $\mathbb{R}^m$ and the method of proof is using the Shadowing properties that Morse-Smale maps have. This subject has been developed in the literature, see [41, 42], and it is clear now that a Morse-Smale map has the Lipschitz Shadowing property and as we will also see, what we call the "Nonautonomous inverse shadowing" property. This last property is going to let us get some results for non autonomous perturbations. One aspect which apparently has not been much developed is the connection between this shadowing properties and distance of attractors, which on the other hand, comes out in a natural way from the definition of shadowing. We exploit this fact to obtain good rates of the distance of attractors. This is accomplished in Chapter 1.

Since our framework for shadowing is finite dimensional, and we eventually want to apply this technique to infinite dimensional systems, say for evolution equations of the type $u_t + Au = F(u)$ in a Hilbert space X, we need a tool that reduces the system to a finite dimensional one. An appropriate tool to accomplish this is Inertial Manifold, which is a smooth finite dimensional manifold positive invariant under the flow and exponentially attractive. In particular the Inertial manifolds contain the attractor if the system has one. These manifolds are constructed as a graph of a smooth function $\Phi : V \to W$ with domain a finite dimensional vector subspace, V and range the orthogonal complement of this space, W which is an infinite dimensional closed subspace. As a matter of fact, we usually have that the space V is generated by m eigenfunctions of the linear operator A. The construction of the manifold is not trivial at all and needs a so called "gap condition" for the eigenvalues of A. This is a major drawback of the theory, since it is not usually obtained. Nevertheless, for some evolutionary equations in a one dimensional domain this gap is shown to exists and it opens the possibility to use this technique. Since we want to study the behavior of the systems under perturbations and obtain good rates for the convergence of the attractors, we will need to study the behavior of these inertial manifolds under perturbations of the system. We study this issue in Chapter 2, where we consider the family of systems of the type $u_t + A_\varepsilon u = F_\varepsilon(u)$ in the phase space X_ε and also the system $u_t + A_0 u = F_0(u)$ in the phase space X_0 and obtain good rates of the distance of the inertial manifolds both in the C^0 topology and in the $C^{1,\theta}$- topology. Notice that since the spaces X_ε and X_0 are different, we need a way to pass the information from one space to the other, via some operators $E : X_0 \to X_\varepsilon$ and $M : X_\varepsilon \to X_0$. We will see that if we are able to estimate the distance of the resolvent operators

$$\|A_\varepsilon^{-1} - E \circ A_0^{-1} M\| \leq \tau(\varepsilon)$$

and the nonlinearities

$$\|F_\varepsilon \circ E - E \circ F_0\| \leq \rho(\varepsilon)$$

then the inertial manifolds, which are given as the graph of functions Φ_ε and Φ_0 satisfy

$$\|\Phi_\varepsilon - E\Phi_0\| \leq C(\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)).$$

We refer to the introduction of Chapter 2 and specially both Section 2.1.1 and Section 2.2.1 for a detailed description of the results.

Notice the factor $\log(\tau(\varepsilon))$ which appears in the rate (see also Theorem 2.1.4 and Theorem 2.2.2). We believe this factor appears because of technical reasons, but we have not been able to get rid of it.

The first two chapters are independent one from the other and they could be read in reverse order, Chapter 2 first and then Chapter 1. They set up two techniques that we will need to use to accomplish our main goal in this thesis.

Finally, in Chapter 3, we apply the techniques developed in Chapter 1 and Chapter 2, to address the problem of the distance of the attractors corresponding to a dynamical system generated by a reaction diffusion equation of the type

$$\begin{cases} u_t - \Delta u + \alpha u = f(u) & \text{in} \quad Q_\varepsilon, \\ \frac{\partial u}{\partial \nu_\varepsilon} = 0 & \text{in} \quad \partial Q_\varepsilon, \end{cases}$$

in a thin domain of the type

$$Q_\varepsilon = \{(x, \varepsilon \mathbf{y}) \in \mathbb{R}^d : (x, \mathbf{y}) \in Q\}, \qquad \varepsilon \in (0,1),$$

where Q is a given set in $\mathbb{R}^d$, (for instance $Q = \{(x, \mathbf{y}) : |\mathbf{y}| \leq r(x)\}$ where $r : (0,1) \to \mathbb{R}^+$ which represents a channel with circular cross sections of radious $r(x)$). We will not restrict to this kind of thin channels and actually we will consider thin domain with not necessarily circular cross sections, but this is a very important class of thin domains. Observe that the factor ε in front of the $\mathbf{y}$ coordinate makes Q_ε a domain that approaches the line segment $[0,1]$ as $\varepsilon \to 0$.

The limit problem is the one dimensional system

$$\begin{cases} u_t - \frac{1}{g}(gu_x)_x + \alpha u = f(u) & \text{in} \quad (0,1), \\ u_x(0) = u_x(1) = 0. \end{cases} \tag{0.4.6}$$

see [31, 46] among others.

It is well known [33] that the time one map of the limit system (0.4.6) is a Morse-Smale gradient map. Moreover, the linear elliptic operator associated to the limit problem also is a Sturm-Liouville operator, which guarantees us that we have appropriate gaps in the spectrum. This two facts, allows us to apply inertial manifold techniques to reduce the system to a finite dimensional one (using the results from Chapter 2) and apply the shadowing techniques to estimate the distance of the attractors (Chapter 1). Putting this two techniques together we will be able to obtain the following estimate for the distance of the attractors

$$\text{dist}_{H^1(Q_\varepsilon)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) \le C\varepsilon^{\frac{d+1}{2}} |\log(\varepsilon)|$$

which improves the one obtained in [31]. Moreover, this estimate, apart from the $|\log(\varepsilon)|$ factor (which comes from the $\log(\tau(\varepsilon))$ factor we were referencing above) appears to be optimal.

Finally, we have included at the end two appendix. In Appendix A we describe a collection of known results which establish the relation, mentioned above, between Morse-Smale structures and the so called Lipschitz shadowing properties.

And in Appendix B we present the prove of the relation between Morse-Smale structures and non-autonomous shadowing properties, which although it cannot be found in the literature, it follows very much the line of argument of [35].

Chapter 1

Morse-Smale systems, shadowing and distance of attractors

One of the central question in dynamical systems is to understand how the dynamic properties of a given system behave under perturbations. This problem is not only interesting mathematically but also from the applications point of view: If a system is supposed to model certain phenomena, the parameter appearing in the equations (coefficients, domain, functions, etc) can be obtained just to a certain degree of accuracy. If a prediction is obtained through the mathematical model, it is fundamental to analyze whether small variations in the coefficients affect the prediction.

Shadowing theory begun around the seventies with the aim to study the relationship between trajectories of a given dynamical system and trajectories of a perturbation of it, see the works by D. V. Anosov [3] and R. Bowen, [16]. Most of the cases studied by this theory are those belonging to the following framework: the relation between the behavior of a given dynamical system and a computed simulation of it. This simulated system is really a perturbation of the original one. In fact, it is an approximation of the real system observed. The relation between the dynamics can be studied in both directions. One, taking the perturbed system as the starting point, that is, studying if for every trajectory of the perturbed system there is a trajectory of the original system near it, *(Direct) Shadowing*. Or in the other way around, if for every trajectory of the system there exists a trajectory of the perturbed system which approximate it, *Inverse Shadowing*. With these tools, if a dynamical system has the Shadowing property, then numerical models reflect the global behavior of trajectories of the system.

There is a class of systems which is very "robust" under perturbations and that has strong ties with shadowing, this is the class of Morse-Smale and Morse-Smale gradient systems, for which a definition is included below in this chapter. The intuitive idea of a Morse-Smale gradient system is that it has an attractor and this attractor is very "hyperbolic" in the sense that it is formed only by hyperbolic equilibria and by connections between this equilibria, and moreover, the stable and unstable manifolds have transversal intersections. This hyperbolicity of equilibria and transversality of stable-unstable manifolds guarantee the robustness under perturbations. The concept of Morse-Smale map concept is now well-studied. Its name is due to the american mathematicians M. Morse and S. Smale, the creator of Morse Theory and the person who studied the importance of this theory in smooth dynamics, respectively. Authors as J. Hale, L.T. Magalhães, W. M. Oliva, have contributed to the theory related to the Morse-Smale concept during the last 30 years, see [25] and [28].

There are several results in the literature where a relation between shadowing properties and Morse-Smale structures is established, see for instance [40, 41, 42, 43].

But one subject which has not been very much exploited in the literature is the relation between shadowing properties of maps and distance of attractors for these maps, and this is a central aspect of this chapter.

In Section 1.1, besides recalling several definitions from shadowing theory and Morse-Smale gradient maps, we analyze the relation between Morse-Smale maps and the concept of Lipschtiz shadowing. Actually, taking some results from [41, 42] we will see that any Morse-Smale gradient map has the property of Uniform Lipschitz shadowing. Moreover, this will allow us to estimate the distance of the attractors of two maps which are in a C^1-neighborhood of a given Morse-Smale gradient map, see Proposition 1.1.8. This result be used in Chapter 3 to estimate the distance of attractors of a reaction diffusion equation in a thin domain problem.

In Section 1.2, we address the problem of the distance of attractors when the perturbed system is non-autonomous. To deal with this case, we define the concept of Nonautonomous inverse shadowing, see Definition 1.2.8, which is very much related to the inverse shadowing, and show also that a Morse-Smale gradient like map has this property, see Proposition 1.2.14. Once we have Lipschitz shadowing and Nonautomous inverse shadowing we can estimate the distance of the attractors of a Morse-Smale gradient like map and the pullback attractor of a non autonomous perturbation of it in terms of the distance of the time one maps.

Related to this Chapter, we have two Appendix where we provide the proofs of Proposition 1.1.8 (see Appendix A) and Proposition 1.2.14 (see Appendix B).

1.1. Morse-Smale Maps and Lipschitz Shadowing

In this section we recall the well known concept of Morse-Smale map and introduce several results as the Lipschitz Shadowing properties of this kind of maps, see [41, 42].

We also exploit the Lipschitz Shadowing property to estimate the distance between the attractors of two maps and we also apply these results to Morse-Smale maps.

1.1.1. Definitions

In this section we introduce the most important concepts of the Shadowing theory, useful for our goal. Most of them can be found in [23].

Throughout this section we will denote by X a Banach Space with norm $\|\cdot\|_X$ and $T : X \to X$, a nonlinear map, no necessary continuous or differentiable. We also denote by $|\cdot|$ the norm in $\mathbb{R}^m$.

Definition 1.1.1. *A* **global trajectory** *of the discrete dynamical system generated by the map T, is a sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}} \subset X$ such that, $x_{n+1} = T(x_n)$ for $n \in \mathbb{Z}$.*

A **negative trajectory** *of the map T is a sequence $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-} \subset X$ such that $x_{n+1} = T(x_n)$, for $n \in \mathbb{Z}^-$. A* **positive trajectory** *of the map T is a sequence $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^+} \subset X$ such that $x_{n+1} = T(x_n)$, for $n \in \mathbb{Z}^+$.*

Definition 1.1.2. *Let $\delta \geq 0$ and $\mathbf{N} \in \mathbb{Z}^+$. A* **global $(\boldsymbol{\delta}, \mathbf{N})$-pseudo-trajectory** *of T is a sequence $\mathbf{y} = \{y_n\}_{n\in\mathbb{Z}} \subset X$ with*

$$||y_{n+k} - T^k(y_n)||_X \leq \delta, \quad for \quad |k| \leq \mathbf{N}, \quad with \quad n \in \mathbb{Z}.$$

Similarly we define **negative $(\boldsymbol{\delta}, \mathbf{N})$-pseudo-trajectory** *and* **positive $(\boldsymbol{\delta}, \mathbf{N})$-pseudo-trajectory**.

*In particular, to simplify we will denote a **(δ,1)-pseudo-trajectory** as a **δ-pseudo-trajectory**, $\delta \geq 0$.*

With $K \subset X$, we denote by $Tr(T, K, \delta)$ the set of all global δ-pseudo-trajectories of the map T in K. Similarly we denote by $Tr^+(T, K, \delta)$ (resp. $Tr^-(T, K, \delta)$) the set of all positive (resp. negative) δ-pseudo-trajectories of T in K. Note that a 0-pseudo trajectory is a trajectory and that we always have the following inclusion

$$Tr(T, K, 0) \subset Tr(T, K, \delta).$$

An important class of δ-pseudo-trajectories of a map T is given by sequences $\{y_n\}_{n\in\mathbb{Z}}$ which are trajectories of a sequence of maps S_n, $n \in \mathbb{Z}$, that is $y_{n+1} = S_n y_n$ for all $n \in \mathbb{Z}$, with $S_n : X \to X$, such that $\sup_{x\in X} \|T(x) - S_n(x)\|_X \leq \delta$, $\forall n \in \mathbb{Z}$. This follows directly from the fact that

$$\|T(y_n) - y_{n+1}\|_X = \|T(y_n) - S_n(y_n)\|_X \leq \delta \quad for \quad n \in \mathbb{Z}^-.$$

That is,

$$\bigcup_{\|T-S\|_X \leq \delta} Tr^-(S, X, 0) \subset Tr^-(T, X, \delta).$$

Let us recall now the definition of attractor for a map $T : X \to X$.

Definition 1.1.3. *An attractor for the map $T : X \to X$ is a set $\mathcal{A} \subset X$ which is compact, invariant ($T(\mathcal{A}) = \mathcal{A}$) and attracts bounded sets of X, that is, for each bounded set $V \subset X$ and for all $\eta > 0$, there exists $n = n(\eta, V)$ such that $dist(T^m(V), \mathcal{A}) \leq \eta$ for all $m \geq n$.*

We consider the space, $l^p(X)$, for $1 \leq p < \infty$, of all infinite negative sequences $\{x_n\}_{n\in\mathbb{Z}^-}$ such that

$$\|x\|_{l^p(X)} = \Big(\sum_{j=1}^{\infty} |x_{-j}|^p\Big)^{1/p} < \infty,$$

and $l^\infty(X)$ the Banach space given by the sequences $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-}$ with $x_n \in X$ and $\|x_n\|_X \leq C$ for all $n \in \mathbb{Z}^-$. That is,

$$l^\infty(X) = \{\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-} : x_n \in X, \quad \|x_n\|_X \leq C \quad \forall n \in \mathbb{Z}^-\},$$

with $C > 0$ a constant and the norm

$$\|\mathbf{x}_-\|_{l^\infty(X)} = \sup\{\|x_n\|_X : n \in \mathbb{Z}^-\}.$$

It is well known these spaces with these norms are Banach spaces.

In the analysis below we are going to need to compare the set of δ-pseudo trajectories and the set of trajectories of a map $T : X \to X$ and specially we will have to deal with the "negative" trajectories and pseudo-trajectories. An appropriate concept for this is the concept of "shadowing" in its multiple variants.

Definition 1.1.4. *A negative sequence* $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-}$ $\boldsymbol{\varepsilon}$**-shadows** *a negative sequence* $\mathbf{y}_- = \{y_n\}_{n\in\mathbb{Z}^-}$ *if and only if,*

$$\|\mathbf{x}_- - \mathbf{y}_-\|_{l^\infty(X)} \leq \varepsilon.$$

So, this property is commutative, that is, $\mathbf{x}_-$ *ε-shadows* $\mathbf{y}_-$ *if and only if* $\mathbf{y}_-$ *ε-shadows* $\mathbf{x}_-$

If for a given sequence $\mathbf{y}_- \in l^\infty(X)$ and $\varepsilon > 0$ we define

$$B_\varepsilon(\mathbf{y}_-) = \{\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-} : \|\mathbf{x}_- - \mathbf{y}_-\|_{l^\infty(X)} < \varepsilon\},$$

then, we can write that a negative sequence $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-}$ $\boldsymbol{\varepsilon}$**-shadows** a sequence $\mathbf{y}_- = \{y_n\}_{n\in\mathbb{Z}^-}$ if $\mathbf{x}_- \in B_\varepsilon(\mathbf{y}_-)$.

The following definitions are very important,

Definition 1.1.5. *The map T has the* **Shadowing** *property in $K \subset X$, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that any negative δ-pseudo-trajectory of T in K is ε-shadowed by a negative trajectory of T in X. That is, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$Tr^-(T, K, \delta) \subset B_\varepsilon(Tr^-(T, X, 0)),$$

Definition 1.1.6. *The map T has the* **Lipschitz Shadowing** *property on $K \subset X$, if there exist constants $L, \delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$, any negative δ-pseudo-trajectory of T in K is $(L\delta)$-shadowed by a negative trajectory of T in X, that is,*

$$Tr^-(T, K, \delta) \subset B_{L\delta}(Tr^-(T, X, 0)).$$

Remark 1.1.7. *If a map T has the Lipschitz Shadowing property then T has the Shadowing property.*

We can easily relate the property of Lipschitz Shadowing with obtaining an estimate of the distance of attractors for two maps.

Proposition 1.1.8. *Let $T_1, T_2 : X \to X$ be maps which have global attractors $\mathcal{A}^1$, $\mathcal{A}^2$. Assume $\mathcal{A}^1, \mathcal{A}^2 \subset \mathcal{U} \subset X$, that T_1, T_2 have both the* **Lipschitz Shadowing property** *on $\mathcal{U}$, with parameters L, δ_0 and $\|T_1 - T_2\|_{L^\infty(\mathcal{U},X)} < \delta_0$. Then, we have*

$$dist_H(\mathcal{A}^1, \mathcal{A}^2) \leq L\|T_1 - T_2\|_{L^\infty(\mathcal{U},X)},$$

Proof. Since T_1 has the Lipschitz Shadowing property on $\mathcal{U}$ with parameters L, δ_0, then any negative δ-pseudo-trajectory of T_1 in $\mathcal{U}$, $\delta \leq \delta_0$, is $L\delta$-shadowed by a negative trajectory of T_1, that is,

$$Tr^-(T_1, \mathcal{U}, \delta) \subset B_{L\delta}(Tr^-(T_1, X, 0)).$$

We consider $r \in \mathcal{A}^2$ and let

$$\mathbf{r}_- = \{r_n\}_{n \in \mathbb{Z}^-} = \{{T_2}^n(r) : n \in \mathbb{Z}^-\} \subset \mathcal{A}^2$$

its negative trajectory under the dynamical system generated by T_2, that is,

$$\mathbf{r}_- \in Tr^-(T_2, \mathcal{A}^2, 0).$$

As we have mentioned above, $\mathbf{r}_-$ is a negative δ-pseudo-trajectory of T^1 in $\mathcal{A}^2 \subset \mathcal{U}$, that is $\mathbf{r}_- \in Tr^-(T^1, \mathcal{U}, \delta)$, with $\delta = \|T_1 - T_2\|_{L^\infty(\mathcal{U},X)}$. So, by the Lipschitz Shadowing property there exists $\mathbf{s}_- = \{s_n\}_{n \in \mathbb{Z}^-} \in Tr^-(T^1, X, 0)$ such that,

$$|r_n - s_n| \leq L\delta, \quad \forall n \in \mathbb{Z}^-.$$

Since

$$|s_n| \leq |r_n| + L\delta,$$

we conclude that $\mathbf{s}_-$ is bounded and for this reason $\mathbf{s}_- \in \mathcal{A}^1$. With this

$$d(r, \mathcal{A}^1) \leq L\delta = L\|T_1 - T_2\|_{L^\infty(\mathcal{U},X)},$$

where $d(\cdot, \cdot)$ is the semi distance defined in (0.4.1). Since $r \in \mathcal{A}_2$ has been chosen in an arbitrary way, we have

$$d(\mathcal{A}^2, \mathcal{A}^1) = \sup_{r \in \mathcal{A}_2} d(r, \mathcal{A}^1) \leq L\delta = L\|T_1 - T_2\|_{L^\infty(\mathcal{U},X)} \tag{1.1.1}$$

where L is independent of ε.

Now, with a completely symmetric argument we obtain also that $dist(\mathcal{A}^1, \mathcal{A}^2) \leq L\|T_1 - T_2\|_{L^\infty(\mathcal{U},X)}$, which shows the result

∎

This result tells us that to estimate the distance of attractors, we should look for maps which have the Lipschitz shadowing property and we will see that an appropriate class to have this property is the class of Morse-Smale gradient like maps.

1.1.2. Morse-Smale Maps

The rigid structure of Morse-Smale systems provide to its attractor of good stability properties which are crucial to analyze the behavior of the system under perturbations. We introduce some definitions and concepts, taken mainly from [25].

Let X be a Banach space and let $T \in \mathcal{C}^r(X,X)$, $r \geq 1$, the space of $\mathcal{C}^r$ maps from X to X which are bounded together with their derivatives up to the order $r \geq 1$. Then, we have the following definitions.

Definition 1.1.9. *A fixed point p of a map $T \in \mathcal{C}^0(X,X)$ is a point satisfying $T(p) = p$. Moreover, if $T \in C^1(X,X)$, a fixed point p of T is* **hyperbolic** *if the spectrum of $DT(p)$ does not intersect the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in $\mathbb{C}$.*

Definition 1.1.10. *For any fixed point p of T, we define the* **stable** *and* **unstable sets** *of T at p as follows*

$$W^s(p) = \{u \in X : T^n u \to p \quad \textit{as} \quad n \to \infty\},$$

$$W^u(p) = \{u \in X : T^{-n}u \quad \textit{is defined for} \quad n \geq 0 \quad \textit{and} \quad T^{-n}u \to p \quad \textit{as} \quad n \to \infty\}.$$

With this, for a given neighborhood U of p, we define the **local stable** *and* **local unstable sets** *of T at p as follows*

$$W^s(p,U) = \{u \in W^s(p) : T^n u \in U, \quad \forall\, n \geq 0\},$$

$$W^u(p,U) = \{u \in W^u(p) : T^{-n} u \in U, \quad \forall\, n \geq 0\}.$$

If p is a hyperbolic fixed point of a map $T \in \mathcal{C}^1(X,X)$, the linear subspaces, $\mathbf{S}(p)$ and $\mathbf{U}(p)$, spanned by the eigenvectors of $DT(p)$ corresponding to the eigenvalues with modulus less than 1 and modulus greater than 1, respectively, form a splitting or decomposition of X, such that

$$X = \mathbf{S}(p) \oplus \mathbf{U}(p),$$

and they are $DT(p)$-invariant, i.e.,

$$DT(p)(\mathbf{S}(p)) = \mathbf{S}(T(p)) = \mathbf{S}(p),$$

$$DT(p)(\mathbf{U}(p)) = \mathbf{U}(T(p)) = \mathbf{U}(p).$$

Also there exist constants $C > 0$ and $\lambda_0 \in (0,1)$ with

$$\|DT^{(n)}(p)u\|_X \leq C\lambda_0^n \|u\|_X, \qquad \forall u \in \mathbf{S}(p), \qquad n \geq 0$$

$$\|DT^{(n)}(p)u\|_X \geq C\lambda_0^{-n} \|u\|_X, \qquad \forall u \in \mathbf{U}(p), \qquad n \geq 0.$$

We call C, λ_0 the **hyperbolicity constants** of p, and the linear subspaces $\mathbf{S}(p)$, $\mathbf{U}(p)$ the **hyperbolic structure** on p.

Remark 1.1.11. *Observe that if p is a fixed point of a map $T \in \mathcal{C}^1(X,X)$, then*

$$D(T^n)(p) = D(\underbrace{T \circ T \circ \cdots T}_{n \text{ times}})(p) = DT(\underbrace{T^{n-1}(p)}_{p}) \circ DT(\underbrace{T^{n-2}(p)}_{p}) \circ \cdots \circ DT(p) = (DT(p))^n.$$

Remark 1.1.12. *If the maps T and $DT(q)$, $q \in X$, are one-to-one on X, then $W^s(p)$ and $W^u(p)$ are $\mathcal{C}^r$-manifolds immersed in X. Moreover, let p be a hyperbolic fixed point of T, if the part of the spectrum of $DT(p)$ lying outside the unit circle is composed of a finite set of m eigenvalues, then $W^u(p,U)$ (respectively $W^s(p,U)$) is a $\mathcal{C}^r$-manifold immersed in X of dimension m (respectively of codimension m), see [25], Appendix.*

Definition 1.1.13. *Two manifolds, U and V, are* **transverse** *if, either $U \cap V = \emptyset$ or for any $z \in U \cap V$, the sum of T_zU and T_zV equals X; with T_zU, T_zV the tangent space of the manifolds U, V in the point z.*

Definition 1.1.14. *Let $T \in C^r(X,X)$, $r \geq 1$, be a map with a global attractor $\mathcal{A}$. The* **non-wandering set** *of T is the set of all $z \in \mathcal{A}$ such that, given a neighborhood V of z in $\mathcal{A}$ and a positive integer n_0, there is an $n > n_0$ such that $T^n(V) \cap V \neq \emptyset$.*

The definition of Morse-Smale maps can be found in [25]. Throughout this chapter, we restrict to the class of Morse-Smale gradient like maps, that is, Morse-Smale maps as in [25] without periodic points. To simplify, we will call them Morse-Smale maps. Then, we introduce now the definition of Morse-Smale map that we will apply.

We denote by $\mathcal{KC}^r(X,X)$ the subset of $\mathcal{C}^r(X,X)$, $r \geq 1$, such that,

(i) $T \in \mathcal{KC}^r(X,X)$ implies that T has a global attractor $\mathcal{A}$.

(ii) $\mathcal{A}$ is upper-semicontinuous on $\mathcal{KC}^r(X,X)$.

Definition 1.1.15. *A map $T \in \mathcal{KC}^r(X,X)$ is* **Morse-Smale (gradient like)** *if*

(1) T, $DT(q)$, $q \in \mathcal{N}(\mathcal{A})$, are one-to-one on $\mathcal{A}$, with $\mathcal{N}(\mathcal{A})$ a neighborhood of $\mathcal{A}$. Hence, everything above for the unstable manifold is satisfied.

(2) The non-wandering set is finite and so consists of the fixed points of T.

(3) All fixed points are hyperbolic with finite dimensional unstable manifolds.

(4) $W^s(p_1)$ is tranversal to $W^u(p_2)$ for all p_1, p_2 fixed points of T.

For example,

Let $f : \mathbb{R} \to \mathbb{R}$ be C^2, such that $f(0) = 0$, $f'(0) = 1$, $uf''(u) < 0$ for $u \neq 0$ and

$$\overline{\lim_{u \to \pm\infty}} \frac{f(u)}{u} \leq 0.$$

Consider the problem

$$\begin{cases} u_t = u_{xx} + \varepsilon f(u), & 0 < x < \pi \\ u = 0 & \text{at} \quad x = 0, \pi, \end{cases} \tag{1.1.2}$$

with constant $\varepsilon \geq 0$. Let $T_\varepsilon : H_0^1(0,\pi) \to H_0^1(0,\pi)$ be the time one map. Then, Daniel B. Henry proved in ([33]) that, for every $\varepsilon \notin \{1^2, 2^2, 3^2, ...\}$, T_ε is a C^2 **Morse-Smale (gradient like)** map.

Remark 1.1.16. *One of the most relevant consequence of this concept is that Morse-Smale maps are Structurally Stable. That is, roughly speaking, the properties of the flow in the attractor are topologically equivalent for small C^1 perturbations of the map. We refer to Hale, Magalhaes, Oliva, (see [28]), for more details in this.*

We want now to relate the **Lipschitz shadowing** property with the concept of **Morse-Smale (gradient like)** map. It turns out that there are strong ties of both notions when the Morse-Smale (gradient like) system is finite dimensional, say $\mathbb{R}^m$. As a matter of fact we have the following results

Lemma 1.1.17. *Let $T : \mathbb{R}^m \to \mathbb{R}^m$, be a* **Morse-Smale (gradient like)** *map which has an attractor $\mathcal{A}$. Then T has the Lipschitz Shadowing property on a neighborhood $\mathcal{N}(\mathcal{A})$ of its attractor.*

Moreover, this property is uniform in a C^1 neighborhood of T. That is, we have the following stronger result

Proposition 1.1.18. *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a* **Morse-Smale (gradient like)** *map, which has an attractor $\mathcal{A}$. There exist a neighborhood Θ of T in the $C^1(\mathcal{N}(\mathcal{A}), \mathbb{R}^m)$ topology and numbers L, d_0 such that, for any map $T' \in \Theta$, T' has the Lipschitz Shadowing property on $\mathcal{N}(\mathcal{A})$ with constants L, d_0.*

The proofs of both results can be found in [42], Theorem 2.2.7 and Theorem 2.2.8, respectively.

We have included in Appendix A a detailed and personal proof of these results.

Remark 1.1.19. *The proof of Proposition 1.1.18 relies strongly in the construction of appropriate subbundles of the tangent space in the neighborhood of the attractor which carries over the decomposition given by the stable and unstable linear manifolds, near the equilibria. This construction, originally done in [47], uses the finite dimensionality of the ambient space.*

Taking into account Proposition 1.1.8 and Proposition 1.1.18, we conclude the following

Proposition 1.1.20. *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a* **Morse-Smale (gradient like)** *map which has a global attractor $\mathcal{A}$. Then, there exists a neighborhood Θ of T in the $C^1(\mathcal{N}(\mathcal{A}), \mathbb{R}^m)$ topology so that, for any $T_1, T_2 \in \Theta$ with $\mathcal{A}_1$, $\mathcal{A}_2$ its respective attractors, we have*

$$dist_H(\mathcal{A}_1, \mathcal{A}_2) \leq L\|T_1 - T_2\|_{L^\infty(\mathcal{N}(\mathcal{A}), \mathbb{R}^m)},$$

with L the Lipschitz Shadowing constant from Proposition 1.1.18.

1.2. Morse-Smale Systems and Non-autonomous Shadowing

In this section we consider a Morse-Smale system and a non-autonomous perturbation of it. We analyze what nonauotonomous shadowing properties the Morse-Smale system satisfies and we study the continuity of its attractor with respect to a non-autonomous perturbation having a pullback attractor. Finally, we will also obtain an optimal estimate for the distance of its attractor and pullback attractor, respectively.

1.2.1. Some more definitions

We present some extra shadowing definitions to apply this theory when we have the situation of a non-autonomous perturbation of a Morse-Smale map. Again, we will denote by X a Banach Space with norm $\|\cdot\|_X$ and $T : X \to X$, a nonlinear map. We also denote by $|\cdot|$ the norm in $\mathbb{R}^m$. Moreover, we will consider $\{T_n\}_{n\in\mathbb{Z}}$ a family of mappings such that for each $n \in \mathbb{Z}$, $T_n : X \to X$ is a nonlinear map, no necessary continuous or differentiable.

We first present some basic concepts of non-autonomous dynamic which can be found in [17].

Definition 1.2.1. *A **process** in X is a family of maps $\{S(t, s) : t \geq s\}$ in $C(X, X)$ such that satisfies the following properties*

(1) $S(t, t) = I$, for all $t \in \mathbb{R}$

(2) $S(t, s) = S(t, \tau)S(\tau, s)$, for all $t \geq \tau \geq s$

(3) $(t, s, x) \to S(t, s)x$ is continuous, $t \geq s$, $x \in X$

In a similar way we have the following definition of discrete process

Definition 1.2.2. *A **discrete process** in X is a family of maps $\{T_{n,m} : n, m \in \mathbb{Z}, n \geq m\}$ such that satisfies the following properties*

(1) $T_{n,n} = I$, for all $n \in \mathbb{Z}$

(2) $T_{n,m} = T_{n,k}T_{k,m}$*, for all* $n \geq k \geq m$

Given any sequence $\{T_n\}_{n\in\mathbb{Z}}$ of maps, we can define a corresponding discrete process by

$$T_{n,n} = I \qquad \text{and} \quad T_{n,m} = T_{n-1} \circ T_{n-2} \circ \cdots \circ T_m \quad \text{for} \quad n > m.$$

Conversely, if $\{T_{n,m}\}$ is a discrete process, then it can be derived from the sequence of maps $\{T_n\}$, where $T_n := T_{n+1,n}$.

We introduce the following three definitions only in the continuos case. For the discrete case they are analogous.

Definition 1.2.3. *A time-dependent family of sets* $\mathcal{A}(\cdot)$ ***is invariant*** *under* $S(\cdot,\cdot)$ *if*

$$S(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t) \qquad \textit{for all} \quad t,\tau \in \mathbb{R} \quad \textit{with} \quad t \geq \tau.$$

Definition 1.2.4. *Let* $S(\cdot,\cdot)$ *be a process. Given* $t \in \mathbb{R}$*, a set* $K \subset X$ ***pullback attracts*** *a set* D *at time* t *under* $S(\cdot,\cdot)$ *if*

$$\lim_{s\to-\infty} dist(S(t,s)D, K) = 0. \tag{1.2.1}$$

K ***pullback attracts bounded sets*** *at time* t *if* (1.2.1) *holds for each bounded subset* D *of* X*. A time-dependent family of subsets of* X*,* $K(\cdot)$*, pullback attracts bounded subsets of* X *under* $S(\cdot,\cdot)$ *if* $K(\cdot)$ *pullback attracts bounded sets at time* t *under* $S(\cdot,\cdot)$*, for each* $t \in \mathbb{R}$*.*

With these concepts we can now introduce the concept of pullback attractor

Definition 1.2.5. *A family* $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ *is the* ***pullback attractor*** *for a process* $S(\cdot,\cdot)$ *if*

(1) $\mathcal{A}(t)$ *is compact for each* $t \in \mathbb{R}$

(2) $\mathcal{A}(\cdot)$ *is invariant with respect to* $S(\cdot,\cdot)$

(3) $\mathcal{A}(\cdot)$ *pullback attracts bounded subsets of* X

(4) $\mathcal{A}(\cdot)$ *is the minimal family of closed sets with property (3)*

(5) $\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)$ *is a bounded set in* X

Remark 1.2.6. *Property* (5) *is not usually considered in the definition of pullback attractor. In our case, we will only consider bounded pullback attractors so we have included this property in the definition*

We present now some non-autonomous shadowing concepts

Definition 1.2.7. *A* **global trajectory** *of the discrete evolution process generated by the family of mappings* $\{T_n\}_{n\in\mathbb{Z}}$*, is a sequence* $\mathbf{x} = \{x_n\}_{n\in\mathbb{Z}} \subset X$ *such that,* $x_{n+1} = T_n(x_n)$ *for* $n \in \mathbb{Z}$.

A **negative trajectory** *of the family* $\{T_n\}_{n\in\mathbb{Z}^-}$ *is a sequence* $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-} \subset X$ *such that* $x_{n+1} = T_n(x_n)$*, for* $n \in \mathbb{Z}^-$*. Similarly for a* **positive trajectory**.

Definition 1.2.8. *The map* T *has the* **Non-autonomous Inverse Shadowing** *property on* $K \subset X$ *with parameters* $\alpha, \beta > 0$*, if for any negative trajectory of* T *in* K*,* $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-} \in Tr^-(T, K, 0)$*, and any family of mappings* $\{\varphi_n\}_{n\in\mathbb{Z}^-}$ *such that,*

$$\varphi_n : X \to X, \qquad n \in \mathbb{Z}^-$$

and

$$\|T - \varphi_n\|_\infty = \sup_{x\in X} \|T(x) - \varphi_n(x)\|_X \le \beta, \qquad \forall n \in \mathbb{Z}^-,$$

there exists a negative trajectory of the family $\{\varphi_n\}_{n\in\mathbb{Z}^-}$*,*

$$\mathbf{y}_- = \{y_n\}_{n\in\mathbb{Z}^-} \in Tr^-(\{\varphi_n\}_{n\in\mathbb{Z}^-}, X, 0),$$

such that

$$\|x_n - y_n\|_X \le \alpha\|T - \varphi_n\|_\infty, \quad \forall n \in \mathbb{Z}^-.$$

That is,

$$Tr^-(T, K, 0) \subset \bigcap_{\|T-\varphi_n\|_\infty \le \beta} B_{\alpha\|T-\varphi_n\|_\infty}(Tr^-(\{\varphi_n\}_{n\in\mathbb{Z}^-}, X, 0)).$$

The *Lipschitz Shadowing* property defined in Definition 1.1.6 and this *Non-autonomous Inverse Shadowing* concept can be put together in the following definition.

Definition 1.2.9. *The map* T *has the* **Non-autonomous Bi-Shadowing** *property on* $K \subset X$*, with parameters* $\alpha, \beta > 0$*, if for any* $\mathbf{x}_- = \{x_n\}_{n\in\mathbb{Z}^-} \in Tr^-(T, K, \delta)$ *with* $0 \le \delta \le \beta$ *and any family of mappings* $\{\varphi_n\}_{n\in\mathbb{Z}^-}$*, for each* $n \in \mathbb{Z}^-$*,* $\varphi_n : X \to X$*, such that*

$$\|T - \varphi_n\|_\infty = \sup_{x\in X} \|T(x) - \varphi_n(x)\|_X \le \beta - \delta, \qquad \forall n \in \mathbb{Z}^-,$$

there exists a negative trajectory $\mathbf{y}_- = \{y_n\}_{n\in\mathbb{Z}^-} \in Tr^-(\{\varphi^n\}_{n\in\mathbb{Z}^-}, X, 0)$ *such that*

$$\|x_n - y_n\|_X \le \alpha(\delta + \|T - \varphi_n\|_\infty), \quad \forall n \in \mathbb{Z}^-.$$

That is, we have the following inclusion

$$Tr^-(T, K, \delta) \subset \bigcap_{\|T-\varphi_n\|_\infty \le \beta-\delta} B_{\alpha(\delta+\|T-\varphi_n\|_\infty)}(Tr^-(\{\varphi_n\}_{n\in\mathbb{Z}^-}, X, 0)).$$

Notice that taking, for each $n \in \mathbb{Z}^-$, $\varphi_n = T$ in the above definition we obtain the *Lipschitz Shadowing* property for $\delta = \beta$ and $L = \alpha$. And if we take $\delta = 0$ we recover the *Non-autonomous Inverse Shadowing* property.

See [1] for a definition of Bi-shadowing, (not necessarily non-autonomous)

1.2.2. Upper semicontinuity estimates and Lipschitz shadowing

Now, we are going to see how Lipschitz Shadowing property provides us with some upper semicontinuity properties of the attractor even for non-autonomous perturbations. We will also obtain some estimates on the distance of the attractors.

Let $T : X \to X$ be the time one map of a dynamical system and $\mathcal{A}$ its global attractor. Let $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}}$ be a family of mappings such that, for each $n \in \mathbb{Z}$,

$$T_{\varepsilon,n} : X \to X, \qquad \varepsilon > 0,$$

and these mappings approximate T, that is,

$$\sup_{n\in\mathbb{Z}} \|T_{\varepsilon,n} - T\|_\infty \longrightarrow 0, \qquad \text{as} \quad \varepsilon \to 0.$$

Assume that the family $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}}$ has a pullback attractor $\{\mathcal{A}_\varepsilon(n) : n \in \mathbb{Z}\}$ for $\varepsilon \neq 0$, see Definition 1.2.5. We start studying some upper semicontinuity property of the family $\{\mathcal{A}_\varepsilon(n) : n \in \mathbb{Z}\}$, $\varepsilon \geq 0$. With this purpose we have the following result.

Proposition 1.2.10. *Let $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}^-}$ with $\varepsilon > 0$ and T be the mappings mentioned above. If T has the Lipschitz Shadowing property in a neighborhood of $\{\mathcal{A}_\varepsilon(k) : k \in \mathbb{Z}^-\}$, then we have*

$$d(\mathcal{A}_\varepsilon(0), \mathcal{A}) \leq L \sup_{n\in\mathbb{Z}^-} \|T_{\varepsilon,n} - T\|_\infty,$$

where L is the constant from the Lipschitz Shadowing property of T.

Proof. The proof of this proposition follows similar steps to the one given in Proposition 1.1.8. Since T has the Lipschitz Shadowing property in a neighborhood of $\{\mathcal{A}_\varepsilon(k) : k \in \mathbb{Z}^-\}$, K, there exist constants δ_0, $L > 0$ such that any negative δ-pseudo-trajectory of T in K, with $\delta \leq \delta_0$, is $L\delta$-shadowed by a negative trajectory of T in X, that is,

$$Tr^-(T, K, \delta) \subset B_{L\delta}(Tr^-(T, X, 0)).$$

Take ε small enough such that $\sup\limits_{n\in\mathbb{Z}^-} \{\|T_{\varepsilon,n} - T\|_\infty\} = \delta < \delta_0$. Let $r_0^\varepsilon \in \mathcal{A}_\varepsilon(0)$, with

$$\mathbf{r}_-^{\boldsymbol{\varepsilon}} = \{r_n^\varepsilon\}_{n\in\mathbb{Z}^-} = \{r_{n+1}^\varepsilon = T_{\varepsilon,n}(r_n^\varepsilon) : n \in \mathbb{Z}^-\} \subset \{\mathcal{A}_\varepsilon(n) : n \in \mathbb{Z}^-\} \subset K$$

its negative trajectory under the family of mappings $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}^-}$, which is bounded since $\{\mathcal{A}_\varepsilon(n), n \in \mathbb{Z}\}$ is a bounded set.

But, $\mathbf{r}_-^{\boldsymbol{\varepsilon}}$ is a negative δ-pseudo-trajectory of T, that is $\mathbf{r}_-^{\boldsymbol{\varepsilon}} \in Tr^-(T, K, \delta)$. So, there exist $\mathbf{r}_- \in Tr^-(T, X, 0)$ such that,

$$|r_n^\varepsilon - r_n| \leq L\delta,$$

for all $n \in \mathbb{Z}^-$. Since

$$|r_n| \leq |r_n^\varepsilon| + L\delta,$$

we conclude that $\mathbf{r}_-$ is bounded and for this reason $\mathbf{r}_- \in \mathcal{A}$. With this

$$d(r_0^\varepsilon, \mathcal{A}) \leq L\delta = L \sup_{n\in\mathbb{Z}^-} \|T_\varepsilon^n - T\|_\infty.$$

Since $r_0^\varepsilon \in \mathcal{A}_\varepsilon(0)$ has been chosen in an arbitrary way, we have

$$d(\mathcal{A}_\varepsilon(0), \mathcal{A}) \leq L\delta = L \sup_{n\in\mathbb{Z}^-} \|T_\varepsilon^n - T\|_\infty, \qquad \forall n \in \mathbb{Z}^-.$$

Which shows the result.

■

This proposition has a direct consequence.

Corolary 1.2.11. *Let T and $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}}$ be as in Proposition 1.2.10. If T has the Lipschitz Shadowing property on K, with $\{\mathcal{A}_\varepsilon(n) : n \in \mathbb{Z}_-\} \subset K$, then*

$$\sup_{k\in\mathbb{Z}} d(\mathcal{A}_\varepsilon(k), \mathcal{A}) \leq L \sup_{n\in\mathbb{Z}} \|T_{\varepsilon,n} - T\|_\infty,$$

where L is the constant from the Lipschitz Shadowing property of T.

Proof. From Proposition 1.2.10, following the arguments of its proof, there exist constants $\delta, L > 0$, such that, taking ε small enough, any negative trajectory of the family of maps $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}^-}$, $\mathbf{r}_-^{\boldsymbol{\varepsilon}} = \{r_n^\varepsilon\}_{n\in\mathbb{Z}^-}$, is $L\delta$-shadowed by a negative trajectory of T, $\mathbf{r}_- = \{r_n\}_{n\in\mathbb{Z}^-}$. If we take $\mathbf{r}_-^{\boldsymbol{\varepsilon}} = \{r_n^\varepsilon\}_{n\in\mathbb{Z}^-}$ with $r_0^\varepsilon \in \mathcal{A}_\varepsilon(k)$, $k \in \mathbb{Z}$, then we have,

$$d(\mathcal{A}_\varepsilon(k), \mathcal{A}) \leq L \sup_{n\leq k} \|T_{\varepsilon,n} - T\|_\infty, \qquad \forall k \in \mathbb{Z}.$$

Then, we have the desired result,

$$\sup_{k\in\mathbb{Z}} d(\mathcal{A}_\varepsilon(k), \mathcal{A}) \leq L \sup_{n\in\mathbb{Z}} \|T_{\varepsilon,n} - T\|_\infty,$$

■

Note that this corolary shows the upper semicontinuity properties of the attractor.

1.2.3. Lower Semicontinuity estimates and Non-autonomous inverse shadowing

Now we study how the Non-autonomous Inverse Shadowing property implies the lower semicontinuity of attractors.

Let T, $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}}$ be the map and family of maps described in Section 1.2.2. To achieve the desired estimate

$$dist(\mathcal{A},\mathcal{A}_\varepsilon(0)) \leq C \sup_{n\in\mathbb{Z}^-} \|T-T_{\varepsilon,n}\|_\infty, \qquad \forall n\in\mathbb{Z}^-$$

with C independent of ε, we use the *Non-autonomous Inverse Shadowing* tool motivated by the following result.

Proposition 1.2.12. *Let T and $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}}$, $\varepsilon>0$, be the mappings described in the last section. If T has the* Non-autonomous Inverse Shadowing *property on $\mathcal{A}$ with parameters α and β, then we have*

$$dist(\mathcal{A},\mathcal{A}_\varepsilon(0)) \leq \alpha \sup_{n\in\mathbb{Z}^-} \|T-T_{\varepsilon,n}\|_\infty.$$

Proof. Let $r_0\in\mathcal{A}$ and $\mathbf{r}_- = \{r_n\}_{n\in\mathbb{Z}^-} = \{T^n(r) : n\in\mathbb{Z}^-\}\subset\mathcal{A}$ be its negative trajectory under the dynamical system generated by T. Since T has the *Non-autonomous Inverse Shadowing* property on $\mathcal{A}$ with parameters α and β then,

$$\mathbf{r}_- \in Tr^-(T,\mathcal{A},0) \subset \bigcap_{\|T-T'_n\|_\infty\leq\beta} B_{\alpha\sup_{n\in\mathbb{Z}^-}\|T-T'_n\|_\infty}(Tr^-(\{T'_n\}_{n\in\mathbb{Z}^-},X,0)).$$

Take $\varepsilon_0>0$ small enough and fixed such that $\|T-T_{\varepsilon,n}\|_\infty\leq\beta$ for all $n\in\mathbb{Z}^-$, $0\leq\varepsilon\leq\varepsilon_0$. Then,

$$\mathbf{r}_- \subset B_{\alpha\sup_{n\in\mathbb{Z}^-}\|T-T_{\varepsilon,n}\|_\infty}(Tr^-(\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}^-},X,0)).$$

That is, there exist $\mathbf{r}^{\boldsymbol{\varepsilon}}_- \in Tr^-(\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}^-},X,0)$ such that

$$|r_n - r_n^\varepsilon| \leq \alpha \sup_{n\in\mathbb{Z}^-}\|T-T_{\varepsilon,n}\|_\infty,$$

for all n for which $\mathbf{r}^{\boldsymbol{\varepsilon}}_-$ is defined. Thus $\mathbf{r}^{\boldsymbol{\varepsilon}}_-$ is bounded. For that, $\mathbf{r}^{\boldsymbol{\varepsilon}}_- \in \{\mathcal{A}_\varepsilon(n) : n\in\mathbb{Z}^-\}$ and also we have

$$|r_0 - r_0^\varepsilon| \leq \alpha \sup_{n\in\mathbb{Z}^-}\|T-T_{\varepsilon,n}\|_\infty,$$

with r_0 and r_0^ε the $n=0$ elements of the sequences $\mathbf{r}_-$ and $\mathbf{r}^{\boldsymbol{\varepsilon}}_-$ respectively, $r_0^\varepsilon\in\mathcal{A}_\varepsilon(0)$. That is

$$dist(r_0,\mathcal{A}_\varepsilon(0)) \leq \alpha \sup_{n\in\mathbb{Z}^-}\|T-T_{\varepsilon,n}\|_\infty.$$

Finally, again since $r_0\in\mathcal{A}$ have been chosen in an arbitrary way, we conclude

$$dist(\mathcal{A},\mathcal{A}_\varepsilon(0)) \leq \alpha \sup_{n\in\mathbb{Z}^-}\|T-T_{\varepsilon,n}\|_\infty,$$

and the proof is finished.

■

As in the last section, this proposition has an important corolary.

Corolary 1.2.13. *Let T and $\{T_{\varepsilon,n}\}_{n\in\mathbb{Z}}$, $\varepsilon > 0$, be the mappings described in the last section. If T has the* Non-autonomous Inverse Shadowing *property on $\mathcal{A}$ with parameters α and β, then,*

$$\sup_{k\in\mathbb{Z}} d(\mathcal{A}, \mathcal{A}_\varepsilon(k)) \leq \alpha \sup_{n\in\mathbb{Z}} \|T - T_{\varepsilon,n}\|_\infty,$$

with α independent of ε and k.

Proof. From Proposition 1.2.12 and following the same arguments used in section 1.2.2, if T has the *Non-autonomous Inverse Shadowing* property on $\mathcal{A}$ with parameters α and β, then we have

$$d(\mathcal{A}, \mathcal{A}_\varepsilon(k)) \leq \alpha \sup_{n\leq k} \|T - T_{\varepsilon,n}\|_\infty, \quad \forall k \in \mathbb{Z}.$$

So, we have,

$$\sup_{k\in\mathbb{Z}} d(\mathcal{A}, \mathcal{A}_\varepsilon(k)) \leq \alpha \sup_{n\in\mathbb{Z}} \|T - T_{\varepsilon,n}\|_\infty,$$

as we wanted to prove.

■

This corolary shows the lower semicontinuity property of the attractor under a non-autonomous perturbation.

1.2.4. Distance of Attractors for a Non-autonomous Perturbation

From the results of the previous sections it is clear that we should look for maps which have the Lipschitz shadowing property and the Non-autonomous inverse shadowing property. Notice that Lemma 1.1.17 gives us that any Morse-Smale gradient like map $T : \mathbb{R}^m \to \mathbb{R}^m$ has the Lipschitz shadowing property. We will show in this section that any such map also has the Non-autonomous Inverse Shadowing property.

More precisely we have the following result.

Proposition 1.2.14. *Let $T : \mathbb{R}^m \to \mathbb{R}^m$, be a* **Morse-Smale gradient like** *map which has an attractor $\mathcal{A} \subset \mathbb{R}^m$. Then T has the Non-autonomous Inverse Shadowing property on a neighborhood $\mathcal{N}(\mathcal{A})$ of its attractor for some parameters α and β.*

In order to continue with the ideas of this chapter, we postpone the proof of this result to Appendix B.

We show now the following result.

Theorem 1.2.15. *Let*

$$\dot{x} = f(x), \tag{1.2.2}$$

be a dissipative gradient system with all the equilibrium points hyperbolic and the time one map of the generated dynamical system $\{S_0(t)\}_{t\in\mathbb{R}}$ is Morse-Smale (gradient like), see Definition 2.1.1. We assume that this system has a global attractor $\mathcal{A}$. We perturb the equation (1.2.2) with a non-autonomous term,

$$\dot{x} = f_\varepsilon(x,t), \qquad \varepsilon > 0 \quad \text{and} \quad t \in \mathbb{R}, \tag{1.2.3}$$

such that, if we denote by $\{S_\varepsilon(t,s) : t \geq s \in \mathbb{R}\}$ the evolution process generated by (1.2.3), then

$$\sup_{s\in\mathbb{R}} \|S_\varepsilon(t+s,s)x_0 - S_0(t)x_0\|_{\mathbb{R}^m} \to 0 \qquad \text{as} \quad \varepsilon \to 0,$$

uniformly for $t \in [0,T]$ and $x_0 \in B$, with B any bounded subset of $\mathbb{R}^m$.

Moreover, we assume that, for each ε, with $\varepsilon > 0$ small enough, there exists a pullback attractor $\{\mathcal{A}_\varepsilon(t) : t \in \mathbb{R}\}$ and that,

$$S_0(t), S_\varepsilon(t,s) : \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

Then, we have

$$\sup_{t\in\mathbb{R}} dist_H(\mathcal{A}_\varepsilon(t), \mathcal{A}_0) \leq \mathbf{C} \sup_{z\in\mathbb{R}} \|S_\varepsilon(z+1,z) - S_0(1)\|_\infty,$$

with $\mathbf{C}$ independent of ε and $t \in \mathbb{R}$.

Proof. Let $\{S_0(t)\}_{t\in\mathbb{R}}$ and $\{S_\varepsilon(t,s) : t \geq s, t, s \in \mathbb{R}\}$ be the dynamical system and evolution process related to (1.2.2) and (1.2.3), and $\mathcal{A}_0$, $\{\mathcal{A}_\varepsilon(t) : t \in \mathbb{R}\}$ its global attractor and pullback attractor, respectively. We denote by $T_0 := S_0(1)$ the time one map of S_0, and $T_{\varepsilon,t,n} := S_\varepsilon(t+n+1, t+n)$, for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$, $\varepsilon \in (0, \varepsilon_0)$. Then,

$$T_0, T_{\varepsilon,t,n} : \mathbb{R}^m \to \mathbb{R}^m, \qquad \text{for each} \quad n \in \mathbb{Z}^-, \quad \text{with} \quad \varepsilon > 0,$$

and,

$$\sup_{t\in\mathbb{R}, n\in\mathbb{Z}} \|T_{\varepsilon,t,n} - T_0\|_\infty = \|S_\varepsilon(t+n+1,t+n) - S_0(1)\|_\infty \longrightarrow 0, \qquad \text{as} \quad \varepsilon \to 0.$$

For fixed $t \in \mathbb{R}$, let us denote by $\{\tilde{\mathcal{A}}_{\varepsilon,t}(n)\}_{n\in\mathbb{Z}}$ the discrete pullback attractor related to the discrete process $\{T_{\varepsilon,t,n}\}_{n\in\mathbb{Z}}$. We show first that $\tilde{\mathcal{A}}_{\varepsilon,t}(n) = \mathcal{A}_\varepsilon(t+n)$ for each $n \in \mathbb{Z}$, $t \in \mathbb{R}$ and $\varepsilon > 0$. We will show it for $n = 0$. For the rest the argument is completely similar. For this, consider the bounded set $B = \bigcup_{s\in\mathbb{R}} \mathcal{A}_\varepsilon(s)$, see Definition 1.2.5.

Hence, for all $\delta \geq 0$ there exists $n \in \mathbb{Z}^-$, with $|n|$ large enough, such that $T_{\varepsilon,t,-1} \circ \cdots \circ T_{\varepsilon,t,n+1} \circ T_{\varepsilon,t,n}(B)$ is contained in a δ-neighborhood of $\tilde{\mathcal{A}}_{\varepsilon,t}(0)$. Moreover, we have $\mathcal{A}_\varepsilon(t+n) \subset B$. By the invariance of pullback attractor with respect to the process, this implies that $\mathcal{A}_\varepsilon(t) \subset T_{\varepsilon,t,-1} \circ \cdots \circ T_{\varepsilon,t,n+1} \circ T_{\varepsilon,t,n}(B)$. So, we conclude that, for all $\delta \geq 0$, $\mathcal{A}_\varepsilon(t)$ is contained in a δ-neighborhood of $\tilde{\mathcal{A}}_{\varepsilon,t}(0)$. Since δ is arbitrarily small, then $\mathcal{A}_\varepsilon(t) \subset \tilde{\mathcal{A}}_{\varepsilon,t}(0)$. In a similar way, considering now $B = \bigcup_{n\in\mathbb{Z}} \tilde{\mathcal{A}}_{\varepsilon,t}(n)$ which is also bounded, and taking its evolution under the continuous process $S_\varepsilon(t,s)$, we obtain $\tilde{\mathcal{A}}_{\varepsilon,t}(0) \subset \mathcal{A}_\varepsilon(t)$. Which shows that $\tilde{\mathcal{A}}_{\varepsilon,t}(0) = \mathcal{A}_\varepsilon(t)$. From now on, we will denote the attractors of the discrete process as $\{\mathcal{A}_\varepsilon(t+n)\}_{n\in\mathbb{Z}}$.

On one side, it is known that T_0 is a Morse-Smale map. And on the other side, from section 2.2.1 and 2.2.2 we have that T_0 has the Lipschitz Shadowing property and the Non-autonomous Inverse Shadowing one with parameters α and β. Since we have all the required hypotheses of Propositions 1.2.10 and 1.2.12, we obtain

$$d(\mathcal{A}_\varepsilon(t), \mathcal{A}_0) \leq L \sup_{n\in\mathbb{Z}} \|T_{\varepsilon,n} - T_0\|_\infty,$$

and,

$$d(\mathcal{A}_0, \mathcal{A}_\varepsilon(t)) \leq \alpha \sup_{n\in\mathbb{Z}} \|T_0 - T_{\varepsilon,n}\|_\infty,$$

with L and α the parameter of the Lipschitz Shadowing and Non-autonomous Inverse Shadowing property, independent of ε.

So, for each $t \in \mathbb{R}$, we have,

$$dist_H(\mathcal{A}_\varepsilon(t), \mathcal{A}_0) \leq \mathbf{C} \sup_{n\in\mathbb{Z}^-} \|T_{\varepsilon,t,n} - T_0\|_\infty = \mathbf{C} \sup_{n\in\mathbb{Z}^-} \|S_\varepsilon(t+n+1, t+n) - S_0(1)\|_\infty,$$

with $\mathbf{C} = \max\{L, \alpha\}$ independent of ε and t. Then, we conclude

$$\sup_{t\in\mathbb{R}} dist_H(\mathcal{A}_\varepsilon(t), \mathcal{A}_0) \leq \mathbf{C} \sup_{t\in\mathbb{R}} \sup_{n\in\mathbb{Z}^-} \|S_\varepsilon(t+n+1, t+n) - S_0(1)\|_\infty =$$

$$= \mathbf{C} \sup_{z\in\mathbb{R}} \|S_\varepsilon(z+1, z) - S_0(1)\|_\infty,$$

with $\mathbf{C}$ independent of ε and t, as we wanted to prove.

■

Chapter 2

Inertial Manifolds

Many systems coming from Partial Differential Equations of evolutionary type, enjoy the property of having an **Inertial Manifold**, that is, a finite dimensional manifold which is smooth, positively invariant and exponentially attracting and carries over all the asymptotic dynamic information of the system. All bounded invariant sets (equilibria, periodic orbits, connecting orbits, attractors, etc) lie in this invariant manifold. The existence of these manifolds is proved once we guarantee that the associated linear elliptic operator of the system has large enough gaps in the spectrum and it is obtained through an appropriate fixed point argument. Proving that we have these gaps is one of the major difficulties of the theory, but still there is a class of equations (for instance, one dimensional parabolic equations) for which these inertial manifolds exist and once they exist, we can reduce the system to a finite dimensional one, for which more techniques are available. We refer to [14, 50, 52] for general references on the theory of Inertial manifolds. See also [49] for an accessible introduction to the theory. These inertial manifolds are smooth, see [22]. We also refer to [32, 25, 13, 52, 17, 19] for general references on dynamics of evolutionary equations.

Due to the relevance of these manifolds, the study of their smoothness properties and the analysis of their behavior under perturbations is very important. Identifying the kind of perturbations allowed so that the inertial manifold persists, estimating the distance of the inertial manifolds and analyzing its smoothness is an important task which have implications in the analysis of the dynamics of the equations.

One of the first examples in which an analysis of the persistence of inertial manifolds was carried over was in [31], where the dynamics of a parabolic equation in a thin domain is analyzed. This paper has been one of the main motivations for our work. In the case treated in [31], the limit equation is one-dimensional for which the gap condition is satisfied since the elliptic operator is of Sturm-Liouville type and spectral gaps are known to exist. The inertial manifold of the limiting one-dimensional problem is proved and after an analysis of the continuity of the spectrum under this perturbation, the inertial manifold is lifted to the perturbed 2-dimensional problem in the thin domain. An estimate of the distance of the inertial manifolds is provided, although it is not as sharp as the one we obtain in this chapter. Moreover, it is proved some smoothness aspects of inertial manifolds. Also, some general results on persistence can be found in [14], and also in [34], where the results are more focused on the numerical approximations of the equations. More recently some results on the behavior of these manifolds under perturbation of the domain have appeared [39, 56], although they do not provide estimates on the distance of the manifolds.

In this work we provide estimates on the distance in the C^0 topology and, also, in the $C^{1,\theta}$ topology, between the inertial manifold of a system and the inertial manifold of a perturbation of it. Moreover, we study the smoothness of these inertial manifolds. The systems may have different phase space (so we may apply these techniques to domain perturbation problems) and the distance is estimated in terms of two parameters only, in the C^0 topology case: the distance of the resolvent operators of the elliptic part and the distance of the nonlinearities of the equations, see Theorem 2.1.4; and in terms of three parameters in the $C^{1,\theta}$ topology case: the two mentioned before and the distance of the differentials of the nonlinearities of the equations, see Theorem 2.2.2.

This Chapter is divided in two main sections, Section 2.1 where we show the existence of inertial manifolds and obtain estimates on the distance of the inertial manifolds in the C^0-topology, and Section2.2 where we show that the manifolds are actually $C^{1,\theta}$ and obtain the convergence in this topology. Both sections start with a short introduction and a section which describe the setting and main results of each section. The reader interested in understanding the mains setting and results, may read Section 2.1.1 first and then Section 2.2.1.

We describe now the contents of this chapter.

Section 2.1 is divided in four subsections.

In Section 2.1.1 we introduce the notation, the main hypothesis that we will impose, **(H1)** related to the convergence of the resolvent operators and **(H2)** related to the convergence of the nonlinearities. We also state the main result of this section, Theorem 2.1.4.

In Section 2.1.2 we analyze the behavior of the linear part of the equations. We show the convergence of the spectrum once the resolvent convergence is imposed and obtain different estimates on the linear problems.

In Section 2.1.3 we obtain the existence of the inertial manifolds. To accomplish this task we apply the results from [52].

In Section 2.1.4 using the implicit definition of the inertial manifolds (given as a fixed point of an appropriate functional) and with the estimates of Section 2.1.2 we prove one of the main results of this chapter, Theorem 2.1.4.

Section 2.2 is divided in three subsections.

In Section 2.2.1 we impose some more hypotheses on the nonlinearities**(H2')**, requiring more smoothness and we state our main result in terms of $C^{1,\theta}$ smoothness of the inertial manifolds and estimates of the distance of these manifolds in this topology, Proposition 2.2.1 and Theorem 2.2.2.

In Section 2.2.2 we analyze the smoothness of inertial manifolds by a fixed point method described in [52].

In Section 2.2.3 we study the convergence of inertial manifolds in the $C^{1,\theta}$ topology. We get an estimate for this convergence obtaining a rate of convergence of inertial manifolds in the C^1 topology and applying the smoothness result obtained in 2.2.2. We present here a proof of Theorem 2.2.2

2.1. C^0-convergence of inertial manifolds

In this section we obtain estimates on the distance of inertial manifolds in the C^0 topology for dynamical systems generated by evolutionary parabolic type equations. We consider the situation where the systems are defined in different phase spaces and we estimate the distance in terms of the distance of the resolvent operators of the corresponding elliptic operators and the distance of the nonlinearities of the equations.

2.1.1. Setting of the problem and main results

Let A_0 be a self-adjoint positive linear operator on a separable real Hilbert space X_0 with domain $D(A_0)$, that is $A_0 : D(A_0) \subset X_0 \to X_0$. We denote by X_0^α, with $\alpha \in [0,1]$, the fractional power spaces associated to the operator A_0 and $\|\cdot\|_{X_0^\alpha}$ its norm, defined in the usual way, see for instance [32, 19],

$$\|u\|_{X_0^\alpha} = \|A_0^\alpha u\|_{X_0}, \qquad \forall u \in X_0^\alpha.$$

We consider the following evolutionary problem,

$$(P_0^\varepsilon)\begin{cases} \quad u_t + A_0 u &= F_0^\varepsilon(u), \\ u(0) \in X_0^\alpha, & \end{cases} \tag{2.1.1}$$

with $F_0^\varepsilon : X_0^\alpha \to X_0$ certain nonlinearity which may depend on ε and guaranteeing that we have global existence of solutions. We will usually denote the solutions of (2.1.1) as u_0^ε, where the subindex 0 makes reference that the elliptic operator A_0 is fixed and does not depend on the parameter ε and the super index ε makes reference to the dependence of the nonlinearity on ε.

We also consider the following perturbed problem,

$$(P_\varepsilon)\begin{cases} \quad u_t + A_\varepsilon u &= F_\varepsilon(u), \qquad 0 < \varepsilon \le \varepsilon_0 \\ u(0) \in X_\varepsilon^\alpha, & \end{cases} \tag{2.1.2}$$

where A_ε is also a self-adjoint positive linear operator on a separable real Hilbert space X_ε, that is $A_\varepsilon : D(A_\varepsilon) = X_\varepsilon^1 \subset X_\varepsilon \to X_\varepsilon$, and the nonlinear term $F_\varepsilon : X_\varepsilon^\alpha \to X_\varepsilon$ is another nonlinearity guaranteeing also global existence of solutions of (2.1.2). We will usually denote by u_ε the solutions of (2.1.2). We will impose appropriate hypotheses on the linear operators A_ε, A_0 and the nonlinearities F_ε, F_0^ε such that the solutions of problems (P_ε) and (P_0^ε) are near, as ε tends to 0, in some sense.

Since our aim is to compare different aspects about the asymptotic dynamics of both problems, (2.1.1) and (2.1.2) and these dynamics live in different functional spaces X_0, and X_ε, we will need to compare functions from X_0 and X_ε, (X_0^α and X_ε^α, respectively, with $\alpha \in [0,1)$ fixed above). We refer to [18] for a general reference where comparison of functions, operators (and their spectrum) defined in different spaces are analyzed, specially for problems related to asymptotic dynamics. See also [4, 6] for similar approaches to particular perturbation problems.

We assume the existence of linear continuous operators, E and M, such that,

$$E : X_0 \to X_\varepsilon, \qquad \text{and} \quad M : X_\varepsilon \to X_0,$$

and,

$$E_{|X_0^\alpha} : X_0^\alpha \to X_\varepsilon^\alpha, \qquad \text{and} \quad M_{|X_\varepsilon^\alpha} : X_\varepsilon^\alpha \to X_0^\alpha.$$

Although these operators depend on ε we will not make explicit this dependence. We will assume they are bounded uniform in ε and we assume there is a constant $\kappa \geq 1$, such that

$$\|E\|_{\mathcal{L}(X_0,X_\varepsilon)}, \|M\|_{\mathcal{L}(X_\varepsilon,X_0)} \leq \kappa, \qquad \|E\|_{\mathcal{L}(X_0^\alpha,X_\varepsilon^\alpha)}, \|M\|_{\mathcal{L}(X_\varepsilon^\alpha,X_0^\alpha)} \leq \kappa. \tag{2.1.3}$$

We also assume these operators satisfy the following property,

$$M \circ E = I. \tag{2.1.4}$$

Remark 2.1.1. *First, note that,* (2.1.4) *implies that* E *is injective and* M *is surjective.*

Moreover, for any $u \in X_0^\alpha$ *with* $\alpha \in [0,1)$, (2.1.3) *and* (2.1.4) *imply*

$$\frac{1}{\kappa}\|u\|_{X_0^\alpha} \leq \|Eu\|_{X_\varepsilon^\alpha} \leq \kappa\|u\|_{X_0^\alpha}. \tag{2.1.5}$$

This is obtained directly as follows,

$$\|u\|_{X_0^\alpha} = \|(M \circ E)u\|_{X_0^\alpha} \leq \kappa\|Eu\|_{X_\varepsilon^\alpha} \leq \kappa^2\|u\|_{X_0^\alpha}.$$

We will also assume that the family of operators A_ε, for $0 \leq \varepsilon \leq \varepsilon_0$, have compact resolvent, that is, the resolvent operators are compact for all $\lambda \in \rho(A_\varepsilon)$ where $\rho(A_\varepsilon)$ is the resolvent set of A_ε. This fact, together with the fact that the operators are selfadjoint, implies that its spectrum is discrete real and consists only of eigenvalues, each one with finite multiplicity. Moreover, the fact that A_ε, $0 \leq \varepsilon \leq \varepsilon_0$, is positive implies that its spectrum is positive. So, we denote by $\sigma(A_\varepsilon)$, the spectrum of the operator A_ε, with,

$$\sigma(A_\varepsilon) = \{\lambda_n^\varepsilon\}_{n=1}^\infty, \qquad \text{and} \quad 0 < c \leq \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq ... \leq \lambda_n^\varepsilon \leq ...$$

and we also denote by $\{\varphi_i^\varepsilon\}_{i=1}^\infty$ an associated orthonormal family of eigenfunctions. Observe that the requirement of the operators A_ε being positive can be relaxed to requiring that they are all bounded from below uniformly in the parameter ε. We can always consider the modified operators $A_\varepsilon + cI$ with c a large enough constant to make the modified operators positive. The nonlinear equations (2.1.1) and (2.1.2) would have to be rewritten accordingly.

With respect to the relation between both operators, A_0 and A_ε we will assume the following hypothesis

(H1). *With α the exponent from problems (2.1.1) and (2.1.2), we have*

$$\|A_\varepsilon^{-1} - EA_0^{-1}M\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{2.1.6}$$

Notice in particular that from (2.1.6) we also have that $\|A_\varepsilon^{-1}E - EA_0^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \to 0$ as $\varepsilon \to 0$. Let us define $\tau(\varepsilon)$ as an increasing function of ε such that

$$\|A_\varepsilon^{-1}E - EA_0^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \le \tau(\varepsilon). \tag{2.1.7}$$

With respect to the nonlinearities F_0^ε and F_ε,

(H2). *We assume that the nonlinear terms $F_0^\varepsilon : X_0^\alpha \to X_0$ and $F_\varepsilon : X_\varepsilon^\alpha \to X_\varepsilon$ for $0 < \varepsilon \le \varepsilon_0$, satisfy:*

(a) They are uniformly bounded, that is, there exists a constant $C_F > 0$ independent of ε such that,

$$\|F_0^\varepsilon\|_{L^\infty(X_0^\alpha, X_0)} \le C_F, \qquad \|F_\varepsilon\|_{L^\infty(X_\varepsilon^\alpha, X_\varepsilon)} \le C_F.$$

(b) They are globally Lipschitz on X_0^α and X_ε^α, respectively, with a uniform Lipstichz constant L_F, that is,

$$\|F_0^\varepsilon(u) - F_0^\varepsilon(u')\|_{X_0} \le L_F \|u - u'\|_{X_0^\alpha}, \quad u, u' \in X_0^\alpha$$

$$\|F_\varepsilon(u) - F_\varepsilon(u')\|_{X_\varepsilon} \le L_F \|u - u'\|_{X_\varepsilon^\alpha} \quad u, u' \in X_\varepsilon^\alpha$$

(c) They have a uniformly bounded support in ε: there exists $R > 0$ such that

$$supp(F_0^\varepsilon) \subset \{u \in X_0^\alpha : \|u\|_{X_0^\alpha} \le R\}, \quad supp(F_\varepsilon) \subset \{u \in X_\varepsilon^\alpha : \|u\|_{X_\varepsilon^\alpha} \le R\}.$$

As we will see below, the convergence of the resolvent operators given by hypothesis **(H1)** guarantees the spectral convergence of the operators, that is, the convergence of the eigenvalues and the eigenfunctions (or eigenprojections). This implies in particular that if we have a gap on the eigenvalues of A_0, we will also have, for ε small enough a similar gap for the eigenvalues of A_ε. This fact, together with the uniform estimates on the nonlinerities given by hypothesis **(H2)**, guarantees that we may construct inertial manifolds of the same dimension for all $0 \leq \varepsilon \leq \varepsilon_0$. We will follow the Lyapunov-Perron method, as developed in [52] to obtain these inertial manifolds $\mathcal{M}_0^\varepsilon$, $\mathcal{M}_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$. As a matter of fact, we consider $m \in \mathbb{N}$ such that $\lambda_m^0 < \lambda_{m+1}^0$ and we denote by $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}$ the canonical orthogonal projection onto the eigenfunctions, $\{\varphi_i^\varepsilon\}_{i=1}^m$, corresponding to the first m eigenvalues of the operator A_ε, $0 \leq \varepsilon \leq \varepsilon_0$ and $\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}}$ its orthogonal complement, see (3.4.4) and (3.4.5). For technical reasons, we express any element belonging to the linear subspace $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon)$ as a linear combination of the elements of the following basis

$$\{\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_1^0), \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_2^0), ..., \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_m^0)\}, \qquad \text{for} \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

with $\{\varphi_i^0\}_{i=1}^m$ the eigenfunctions related to the first m eigenvalues of A_0, which will be seen below that is a basis in $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon)$ and in $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon^\alpha)$. We will denote by $\psi_i^\varepsilon = \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_i^0)$.

Let us denote by j_ε the isomorphism from $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon) = [\psi_1^\varepsilon, ..., \psi_m^\varepsilon]$ onto $\mathbb{R}^m$, that gives us the coordinates of each vector. That is,

$$\begin{aligned} j_\varepsilon : \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(X_\varepsilon) &\longrightarrow \mathbb{R}^m, \\ w_\varepsilon &\longmapsto z, \end{aligned} \tag{2.1.8}$$

where $w_\varepsilon = \sum_{i=1}^m z_i \psi_i^\varepsilon$ and $z = (z_1, ..., z_m)$.

We denote by $|\cdot|$ the usual euclidean norm in $\mathbb{R}^m$,

$$|z| = \left(\sum_{i=1}^m z_i^2 \right)^{\frac{1}{2}}, \tag{2.1.9}$$

and by $|\cdot|_{\varepsilon,\alpha}$ for $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \alpha \leq 1$, the following one,

$$|z|_{\varepsilon,\alpha} = \left(\sum_{i=1}^m z_i^2 (\lambda_i^\varepsilon)^{2\alpha} \right)^{\frac{1}{2}}. \tag{2.1.10}$$

and observe that $|\cdot|_{\varepsilon,0} = |\cdot|$.

We consider the spaces $(\mathbb{R}^m, |\cdot|)$ and $(\mathbb{R}^m, |\cdot|_{\varepsilon,\alpha})$, that is, $\mathbb{R}^m$ with the norm $|\cdot|$ and $|\cdot|_{\varepsilon,\alpha}$, respectively, and notice that for $w_0 = \sum_{i=1}^m p_i \psi_i^0$ and $0 \leq \alpha < 1$ we have that,

$$\|w_0\|_{X_0^\alpha} = |j_0(w_0)|_{0,\alpha}. \tag{2.1.11}$$

As we mentioned in the introduction, we are looking for inertial manifolds for systems (2.1.1) and (2.1.2). That is, finite dimensional manifolds which are smooth, invariant and exponentially attracting and carry over all the asymptotic dynamic information of the systems. These manifolds will be obtained as graphs of appropriate functions. This motivates the introduction of the family of set $\mathcal{F}_\varepsilon(L,\rho)$,

$$\mathcal{F}_\varepsilon(L,\rho) = \{\chi_\varepsilon : \mathbb{R}^m \to \mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(X^\alpha_\varepsilon), \quad \text{such that} \quad \text{supp}\ \chi_\varepsilon \subset \{|z|_{\varepsilon,\alpha} < \rho\} \quad \text{and}$$

$$\|\chi_\varepsilon(z) - \chi_\varepsilon(z')\|_{X^\alpha_\varepsilon} \le L|z - z'|_{\varepsilon,\alpha} \quad z, z' \in \mathbb{R}^m\}. \tag{2.1.12}$$

Then we can show the following result.

Proposition 2.1.2. *Let hypotheses* **(H1)** *and* **(H2)** *be satisfied. Assume also that* $m \ge 1$ *is such that,*

$$\lambda^0_{m+1} - \lambda^0_m \ge 3(\kappa+2)L_F\left[(\lambda^0_m)^\alpha + (\lambda^0_{m+1})^\alpha\right], \tag{2.1.13}$$

and

$$(\lambda^0_m)^{1-\alpha} \ge 6(\kappa+2)L_F(1-\alpha)^{-1}. \tag{2.1.14}$$

with κ *the bound of operators* E *and* M, *see* (2.1.3).

Then, there exist $L < 1$ *and* $\varepsilon_0 > 0$ *such that there exist an inertial manifold* $\mathcal{M}^\varepsilon_0$ *for (2.1.1) and* $\mathcal{M}_\varepsilon$ *for (2.1.2), for all* $0 < \varepsilon \le \varepsilon_0$, *given by the "graph" of a function* $\Phi^\varepsilon_0 \in \mathcal{F}_0(L,R)$, $\Phi_\varepsilon \in \mathcal{F}_\varepsilon(L,R)$, *respectively, where* R *is the one given by hypothesis* **(H2)** *(c).*

The proof of this result can be found in Section 2.1.3, which is based on [52], Theorem 81.1.

Remark 2.1.3. *i) Observe that the gap condition is stated for the eigenvalues of the limit problem. In particular, this implies that the inertial manifold is obtained of the same dimension* m *for all values of the parameter* $0 \le \varepsilon \le \varepsilon_0$.
ii) We have written quotations in the word "graph" since the manifolds $\mathcal{M}^\varepsilon_0$, $\mathcal{M}_\varepsilon$ *are not properly speaking the graphs of the functions* Φ^ε_0, Φ_ε *but rather the graphs of the appropriate functions obtained via the isomorphisms* j_0, j_ε *which identify* $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(X^\alpha_\varepsilon)$ *with* $\mathbb{R}^m$, $0 \le \varepsilon \le \varepsilon_0$. *That is,*

$$\mathcal{M}^\varepsilon_0 = \{j_0^{-1}(z) + \Phi^\varepsilon_0(z); \quad z \in \mathbb{R}^m\},$$

and

$$\mathcal{M}_\varepsilon = \{j_\varepsilon^{-1}(z) + \Phi_\varepsilon(z); \quad z \in \mathbb{R}^m\}$$

The main result we want to show in this section is the following:

Theorem 2.1.4. *Let hypotheses* **(H1)**, **(H2)** *and gap conditions* (2.1.13), (2.1.14) *be satisfied, so that Proposition 2.1.2 hold and we have inertial manifolds* $\mathcal{M}_0^\varepsilon$ *and* $\mathcal{M}^\varepsilon$ *given as the graphs of the functions* Φ_0^ε *and* Φ_ε. *If we denote*

$$\rho(\varepsilon) = \sup_{u \in \mathcal{M}_0^\varepsilon} \|F_\varepsilon\big(Eu\big) - EF_0^\varepsilon\big(u\big)\|_{X_\varepsilon}, \tag{2.1.15}$$

then we have,

$$\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} \leq C\Big(\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)\Big), \tag{2.1.16}$$

where $\tau(\varepsilon)$ *is defined by* (2.1.7) *and* C *a constant independent of* ε.

Remark 2.1.5. *Observe that the estimate* (2.1.16) *consists of the terms,* $\tau(\varepsilon)|\log(\tau(\varepsilon))|$, *inherited from the distance of the resolvent operators and* $\rho(\varepsilon)$ *inherited from the distance of the nonlinear terms. The factor* $|\log(\tau(\varepsilon))|$ *seems to appear because of technical reasons. A better estimates would be* $\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} \leq C[\tau(\varepsilon) + \rho(\varepsilon)]$, *which we have not been able to show, although it is very plausible that this would be true and it should be the optimal rate.*

2.1.2. Linear analysis and spectral behavior

The spectral decomposition of the operator A_ε implies that if $\lambda \in \rho(A_\varepsilon)$ then,

$$(\lambda - A_\varepsilon)^{-1}u = \sum_{i=1}^{\infty} \frac{1}{\lambda - \lambda_i^\varepsilon}(u, \varphi_i^\varepsilon)\varphi_i^\varepsilon.$$

In particular, for $\varepsilon \geq 0$,

$$\|(\lambda - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq \max_{i \in \mathbb{N}} \left\{ \frac{1}{|\lambda - \lambda_i^\varepsilon|}, \quad \lambda_i^\varepsilon \in \sigma(A_\varepsilon) \right\} = \frac{1}{dist(\lambda, \sigma(A_\varepsilon))}.$$

For $\alpha \geq 0$ and for all $0 \leq \varepsilon \leq \varepsilon_0$, let $A_{\varepsilon|_{X_\varepsilon^\alpha}} : X_\varepsilon^{1+\alpha} \subset X_\varepsilon^\alpha \to X_\varepsilon^\alpha$, with domain $X_\varepsilon^{1+\alpha} \subset X_\varepsilon^1$, be the restriction of A_ε to the fractional power space $X_\varepsilon^\alpha \subset X_\varepsilon$ so that,

$$A_\varepsilon u = A_{\varepsilon|_{X_\varepsilon^\alpha}} u \qquad \forall u \in X_\varepsilon^{1+\alpha}.$$

Then $A_{\varepsilon|_{X_\varepsilon^\alpha}}$ is also a sectorial operator on X_ε^α and with a similar spectral decomposition as above, we can also obtain the estimate

$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon^\alpha)} \leq \frac{1}{dist(\lambda, \sigma(A_\varepsilon))}, \qquad 0 \leq \varepsilon \leq \varepsilon_0.$$

Note that, for $\lambda \in \rho(-A_\varepsilon)$ and $\alpha \geq 0$,

$$\|(\lambda I + A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon^\alpha)} \leq \frac{1}{dist(\lambda, \sigma(-A_\varepsilon))}, \qquad 0 \leq \varepsilon \leq \varepsilon_0.$$

Moreover, since A_ε is a sectorial operator, $-A_\varepsilon$ is the infinitesimal generator of a linear semigroup that we denote as $e^{-A_\varepsilon t}$, where,

$$e^{-A_\varepsilon t} = \frac{1}{2\pi i}\int_\Gamma (\lambda I + A_\varepsilon)^{-1} e^{\lambda t} d\lambda,$$

with Γ a contour in the resolvent set of $-A_\varepsilon$, $\rho(-A_\varepsilon)$, with $arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$, (see [32]). Since A_ε, $\varepsilon \geq 0$, is a self-adjoint operator, the formula above is equivalent to

$$e^{-A_\varepsilon t}u = \sum_{i=1}^{\infty} e^{-\lambda_i^\varepsilon t}(u, \varphi_i^\varepsilon)\varphi_i^\varepsilon. \tag{2.1.17}$$

Moreover, we have the following result.

Lemma 2.1.6. *We have the following estimates for the linear semigroup*

$$\|e^{-A_\varepsilon t}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq e^{-\lambda_1^\varepsilon t} \leq 1,$$

and,

$$\|e^{-A_\varepsilon t}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \leq e^{-\lambda_1^\varepsilon t}\left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha,$$

for $t \geq 0$.

Proof. With the expression of the semigroup given by (2.1.17), we get

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} = \left(\sum_{i=1}^{\infty} e^{-2\lambda_i^\varepsilon t}(u, \varphi_i^\varepsilon)^2(\lambda_i^\varepsilon)^{2\alpha}\right)^{\frac{1}{2}}.$$

The function $f(\lambda) = e^{-\lambda t}\lambda^\alpha$ attains its maximum at $\lambda = \frac{\alpha}{t}$. Then, we have to distinguish two cases:

If $\frac{\alpha}{t} < \lambda_1^\varepsilon$, we obtain

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} \leq e^{-\lambda_1^\varepsilon t}(\lambda_1^\varepsilon)^\alpha \|u\|_{X_\varepsilon}.$$

And if $\lambda_1^\varepsilon \leq \frac{\alpha}{t}$,

$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} \leq e^{-\alpha}\left(\frac{\alpha}{t}\right)^\alpha \|u\|_{X_\varepsilon} \leq e^{-\lambda_1^\varepsilon t}\left(\frac{\alpha}{t}\right)^\alpha \|u\|_{X_\varepsilon}.$$

That is,
$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon^\alpha} \leq e^{-\lambda_1^\varepsilon t}\left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha \|u\|_{X_\varepsilon}.$$
In the same way, since
$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon} = \left(\sum_{i=1}^\infty e^{-2\lambda_i^\varepsilon t}(u,\varphi_i^\varepsilon)^2\right)^{\frac{1}{2}},$$
then, we obtain,
$$\|e^{-A_\varepsilon t}u\|_{X_\varepsilon} \leq e^{-\lambda_1^\varepsilon t}\|u\|_{X_\varepsilon}.$$
This concludes the proof of the result. ■

With respect to the relation of the spectrum we have the following result.

Lemma 2.1.7. *If K_0 is a compact set of the complex plane with $K_0 \subset \rho(-A_0)$, the resolvent set of $-A_0$, and hypothesis* **(H1)** *is satisfied, then there exists $\varepsilon_0(K_0) > 0$ such that $K_0 \subset \rho(-A_\varepsilon)$ for all $0 < \varepsilon \leq \varepsilon_0(K_0)$. Moreover, we have the estimates:*
$$\|(\lambda I + A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \leq C(K_0), \qquad \|(\lambda I + A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq C(K_0), \quad (2.1.18)$$
for all $\lambda \in K_0$, $0 < \varepsilon \leq \varepsilon_0(K_0)$.

Proof. Let us start by showing the following: if $\lambda_{\varepsilon_n} \in \rho(-A_{\varepsilon_n})$ with $\|(\lambda_{\varepsilon_n} I + A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X_{\varepsilon_n}^\alpha)} \geq k_n$, $k_n \to +\infty$ as $n \to +\infty$, and $\lambda_{\varepsilon_n} \to \lambda_0$, then $\lambda_0 \in \sigma(-A_0)$.

Then, assume there exists a sequence $\{\lambda_{\varepsilon_n}\} \in \rho(-A_{\varepsilon_n})$ with
$$\|(\lambda_{\varepsilon_n} I + A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X_{\varepsilon_n}^\alpha)} \geq k_n,$$
and such that $\lambda_{\varepsilon_n} \to \lambda_0$ as $\varepsilon_n \to 0$, for some λ_0. This implies that there exists $f_{\varepsilon_n} \in X_{\varepsilon_n}$ with $\|f_{\varepsilon_n}\|_{X_{\varepsilon_n}} = 1$ and if $w_{\varepsilon_n} = (\lambda_{\varepsilon_n} I + A_{\varepsilon_n})^{-1} f_{\varepsilon_n}$, then $\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha} \to +\infty$.

If we define $u_{\varepsilon_n} = w_{\varepsilon_n}/\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}$, then $\lambda_{\varepsilon_n} u_{\varepsilon_n} + A_{\varepsilon_n} u_{\varepsilon_n} = f_{\varepsilon_n}/\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}$, which implies
$$-A_{\varepsilon_n} u_{\varepsilon_n} = \lambda_{\varepsilon_n} u_{\varepsilon_n} - \frac{f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}.$$
Let $\hat{u}_{\varepsilon_n} \in X_0^\alpha$ satisfy the following equation,
$$-A_0 \hat{u}_{\varepsilon_n} = \lambda_{\varepsilon_n} M u_{\varepsilon_n} - \frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}. \quad (2.1.19)$$
If we study the norm of the right side, since $\left\|\frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}\right\|_{X_0} \to 0$, we have, by (2.1.3)
$$\left\|\lambda_{\varepsilon_n} M u_{\varepsilon_n} - \frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}\right\|_{X_0} \leq \kappa|\lambda_{\varepsilon_n}|\|u_{\varepsilon_n}\|_{X_{\varepsilon_n}} + \left\|\frac{M f_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}}\right\|_{X_0} \leq C.$$

So, $\{\hat{u}_{\varepsilon_n}\} \subset X_0^\alpha$ is a compact family. Then, there exists a $\hat{u}_0 \in X_0^\alpha$ and a subsequence, we denote it again as $\hat{u}_{\varepsilon_n}$, such that $\hat{u}_{\varepsilon_n} \to \hat{u}_0$ in X_0^α, as $\varepsilon_n \to 0$. Moreover, by hypothesis **(H1)**, we have, $\|u_{\varepsilon_n} - E\hat{u}_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha} \to 0$. And,

$$\|u_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}} \leq \|u_{\varepsilon_n} - E\hat{u}_{\varepsilon_n}\|_{X_{\varepsilon_n}} + \|E\hat{u}_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}} \leq$$

$$\leq \|u_{\varepsilon_n} - E\hat{u}_{\varepsilon_n}\|_{X_{\varepsilon_n}} + \kappa\|\hat{u}_{\varepsilon_n} - \hat{u}_0\|_{X_0} \to 0.$$

So, again by (2.1.3),

$$\|Mu_{\varepsilon_n} - \hat{u}_0\|_{X_0} = \|M(u_{\varepsilon_n} - E\hat{u}_0)\|_{X_0} \leq \kappa\|u_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}} \to 0.$$

Hence, via subsequences, $\lambda_{\varepsilon_n} Mu_{\varepsilon_n} - \frac{Mf_{\varepsilon_n}}{\|w_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha}} \to \lambda_0\hat{u}_0$ in X_0 for some $\hat{u}_0 \in X_0^\alpha$. Also, from the definition of u_{ε_n} we have that $\|u_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha} = 1$. Then $1 = \|u_{\varepsilon_n}\|_{X_{\varepsilon_n}^\alpha} \leq \|u_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}^\alpha} + \|E\hat{u}_0\|_{X_{\varepsilon_n}^\alpha} \leq \|u_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}^\alpha} + \kappa\|\hat{u}_0\|_{X_0^\alpha}$. But since $\|u_{\varepsilon_n} - E\hat{u}_0\|_{X_{\varepsilon_n}^\alpha} \to 0$ then $\|\hat{u}_0\|_{X_0^\alpha} > 0$ and hence $\hat{u}_0 \neq 0$. So, from equation (2.1.19) and the above estimates, we obtain $-A_0\hat{u}_0 = \lambda_0\hat{u}_0$, which shows that $\lambda_0 \in \sigma(-A_0)$.

Next, we apply this result to prove our lemma. For the first part, we proceed as follows. If $K_0 \cap \sigma(-A_\varepsilon)$ is non empty for ε small enough, then there exists a sequence $\varepsilon_n \to 0$ and $\hat{\lambda}_{\varepsilon_n} \in K_0 \cap \sigma(-A_{\varepsilon_n})$. Since the spectrum of $-A_{\varepsilon_n}$ is discrete for all ε_n, for each n we can choose $\lambda_{\varepsilon_n} \in \rho(-A_{\varepsilon_n})$ such that $|\lambda_{\varepsilon_n} - \hat{\lambda}_{\varepsilon_n}| < \frac{1}{n}$ and $\|(\lambda_{\varepsilon_n} I + A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X_{\varepsilon_n}^\alpha)} > k_n$ with $k_n \to +\infty$. Moreover, since K_0 is compact, there is a subsequence $\hat{\lambda}_{\hat{\varepsilon}_n}$ with $\hat{\lambda}_{\hat{\varepsilon}_n} \to \lambda_0$ and $\lambda_0 \in K_0$. Then, we have just proved that, $\lambda_0 \in \sigma(-A_0)$. This is a contradiction. So, $K_0 \cap \sigma(-A_\varepsilon)$ is empty, and then $K_0 \subset \rho(-A_\varepsilon)$ as we wanted to prove.

To obtain the desired estimates, suppose there exist sequences $\{\lambda_n\} \in K_0$ and $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to +\infty$ such that,

$$\|(\lambda_n I + A_{\varepsilon_n})^{-1}\|_{\mathcal{L}(X_{\varepsilon_n}, X_{\varepsilon_n}^\alpha)} \geq k_n,$$

with $k_n \to +\infty$. Since K_0 is a compact set, there exists a $\lambda_0 \in K_0$ and a subsequence $\{\lambda_{n_k}\} \in K_0$ with $\lambda_{n_k} \to \lambda_0$, $\lambda_0 \in K_0$, and

$$\|(\lambda_{n_k} I + A_{\varepsilon_{n_k}})^{-1}\|_{\mathcal{L}(X_{\varepsilon_{n_k}}, X_{\varepsilon_{n_k}}^\alpha)} \geq k_{n_k}.$$

Then, we have proved above that, $\lambda_0 \in \sigma(-A_0)$. This is a contradiction because $\lambda_0 \in K_0 \subset \rho(-A_0)$. So, we have for $\lambda \in K_0$,

$$\|(\lambda I + A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \leq C(K_0), \qquad \|(\lambda I + A_\varepsilon)^{-1}\|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq C(K_0).$$

This concludes the proof. ■

Remark 2.1.8. *The result just proved implies the uppersemicontinuity of the spectrum: if $\lambda_\varepsilon \in \sigma(A_\varepsilon)$ and $\lambda_\varepsilon \to \lambda_0$ (via subsequences) then $\lambda_0 \in \sigma(A_0)$.*

Now we want to estimate $\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}$. We have the following result.

Lemma 2.1.9. *With the notation above and assuming hypothesis* **(H1)** *is satisfied, if $\lambda \in \rho(-A_0)$ and ε is small enough so that $\lambda \in \rho(-A_\varepsilon)$, we have*

$$\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq C_3^\varepsilon(\lambda)\tau(\varepsilon),$$

where $C_3^\varepsilon(\lambda) = \left(1 + \frac{|\lambda|}{dist(\lambda, \sigma(-A_\varepsilon))}\right)\left(1 + \frac{|\lambda|}{dist(\lambda, \sigma(-A_0))}\right)$ *and $\tau(\varepsilon)$ is defined by* (2.1.7).

Proof. First of all notice that from Lemma 2.1.7 if $\lambda \in \rho(-A_0)$ then $\lambda \in \rho(-A_\varepsilon)$ for ε small enough. Hence $(\lambda I + A_\varepsilon)^{-1}$ and $(\lambda I + A_0)^{-1}$ are well defined for all $\lambda \in \rho(-A_0)$.

We are interested in estimating,

$$\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}.$$

The first thing we are going to do is to show the following identity:

$$(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1} = [I - (\lambda I + A_\varepsilon)^{-1}\lambda](A_\varepsilon^{-1}E - EA_0^{-1})[I - \lambda(\lambda I + A_0)^{-1}]. \tag{2.1.20}$$

First, note that

$$(I + A_\varepsilon^{-1}\lambda)[I - (A_\varepsilon + \lambda I)^{-1}\lambda] = I, \tag{2.1.21}$$

then,

$$(I + A_\varepsilon^{-1}\lambda)(\lambda I + A_\varepsilon)^{-1} = A_\varepsilon^{-1}.$$

Hence,

$$\begin{gathered}(I + A_\varepsilon^{-1}\lambda)\left[(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right] = \\ = A_\varepsilon^{-1}E - E(\lambda I + A_0)^{-1} - A_\varepsilon^{-1}\lambda E(\lambda I + A_0)^{-1}.\end{gathered}$$

Since,

$$\begin{gathered}E(\lambda I + A_0)^{-1} = EA_0^{-1} - EA_0^{-1} + E(\lambda I + A_0)^{-1} = EA_0^{-1} - EA_0^{-1}[I - A_0(\lambda I + A_0)^{-1}] = \\ = EA_0^{-1} - EA_0^{-1}[(A_0 + \lambda I)^{-1}\lambda],\end{gathered}$$

we have,

$$\begin{gathered}(I + A_\varepsilon^{-1}\lambda)\left[(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right] = \\ = A_\varepsilon^{-1}E - A_\varepsilon^{-1}E\lambda(\lambda I + A_0)^{-1} - EA_0^{-1} + EA_0^{-1}[(A_0 + \lambda I)^{-1}\lambda] = \\ = (A_\varepsilon^{-1}E - EA_0^{-1})[I - \lambda(\lambda I + A_0)^{-1}].\end{gathered}$$

By (2.1.21), $[I - \lambda(A_\varepsilon + \lambda I)^{-1}](I + A_\varepsilon^{-1}\lambda) = I$, then we obtain the desired identity (2.1.20),

$$(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1} = [I - \lambda(A_\varepsilon + \lambda I)^{-1}](A_\varepsilon^{-1}E - EA_0^{-1})[I - \lambda(\lambda I + A_0)^{-1}].$$

Hence, since hypothesis **(H1)** is satisfied, we obtain the desired estimates,

$$\|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \leq$$

$$\leq \|(I-\lambda(A_\varepsilon+\lambda I)^{-1}\|_{\mathcal{L}(X_\varepsilon^\alpha,X_\varepsilon^\alpha)}\|A_\varepsilon^{-1}E-EA_0^{-1}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)}\|I-\lambda(\lambda I+A_0)^{-1}\|_{\mathcal{L}(X_0,X_0)} \leq$$

$$\leq \left(1+\frac{|\lambda|}{dist(\lambda,\sigma(-A_\varepsilon))}\right)\tau(\varepsilon)\left(1+\frac{|\lambda|}{dist(\lambda,\sigma(-A_0))}\right).$$

This concludes the proof. ■

We can easily show now,

Corolary 2.1.10. *(i) If $K_0 \subset \rho(-A_0)$ as in Lemma 2.1.7 and $\Sigma_{-a,\phi}$ is the set of the complex plane described by*

$$\Sigma_{-a,\phi} = \{\lambda \in \mathbb{C} : |arg(\lambda + a)| \leq \pi - \phi\},$$

with $a \geq 0$, then,

$$\sup_{\lambda\in K_0\cup\Sigma_{-a,\phi}} C_3^\varepsilon(\lambda) \leq C_3,$$

for some constant C_3 independent of ε.

(ii) If we take $a = 0$ and $\phi = \frac{\pi}{4}$ then

$$C_3^\varepsilon(\lambda) \leq \left(1+\frac{1}{\sin(\phi)}\right)^2 \leq 6, \qquad \textit{for all} \quad \lambda \in \Sigma_{0,\frac{\pi}{4}}. \tag{2.1.22}$$

Remark 2.1.11. *Note that, although $C_3^\varepsilon(\lambda)$ depends on ε, thanks to the uppersemicontinuity of the eigenvalues, see Remark 2.1.8, we can consider it uniform in ε.*

The estimate found in Lemma 2.1.9 will be applied to obtain estimates on the distance of the spectral projections and estimates on the distance of the linear semigroups generated by A_0 and A_ε. Let us start with the spectral projections.

Let us assume that for some $m = 1, 2, ...$ we have $\lambda_m^0 < \lambda_{m+1}^0$ and as we have mentioned in the introduction, we denote by $\{\varphi_i^\varepsilon\}_{i=1}^m$ the first m eigenfunctions of the operator A_ε, $0 \leq \varepsilon \leq \varepsilon_0$ and by $\mathbf{P_m^\varepsilon}$ the canonical orthogonal projection onto the subspace $[\varphi_1^\varepsilon, \ldots, \varphi_m^\varepsilon]$, that is, if $0 < \varepsilon \leq \varepsilon_0$

$$\begin{aligned} \mathbf{P_m^\varepsilon} : X_\varepsilon &\longrightarrow X_\varepsilon \\ v &\longrightarrow \mathbf{P_m^\varepsilon}(v) = \textstyle\sum_{i=1}^m (v, \varphi_i^\varepsilon)_{X_\varepsilon}\varphi_i^\varepsilon \end{aligned} \tag{2.1.23}$$

or if $\varepsilon = 0$,

$$\begin{aligned} \mathbf{P_m^0} : X_0 &\longrightarrow X_0 \\ v &\longrightarrow \mathbf{P_m^0}(v) = \textstyle\sum_{i=1}^m (v, \varphi_i^0)_{X_0}\varphi_i^0 \end{aligned} \tag{2.1.24}$$

Notice that in a natural way, the projections may be defined in the intermediate space X^α_ε and, since it is a finite linear combination of eigenfunctions, its range is contained also in X^α_ε.

We have the following estimate.

Lemma 2.1.12. *Let $\{\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}\}_{0\leq\varepsilon\leq\varepsilon_0}$ be the family of canonical orthogonal projections described above, $v \in X_0$, Γ a curve in the complex plane contained in $\rho(-A_0)$ and encircling the first m eigenvalues of $-A_0$. Then if we assume* **(H1)** *is satisfied, we have*

$$\|\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E(v) - E\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}(v)\|_{X^\alpha_\varepsilon} \leq C_P\tau(\varepsilon)\|v\|_{X_0},$$

with $C_P = \frac{|\Gamma|}{2\pi}\sup_{\lambda\in\Gamma} C^\varepsilon_3(\lambda)$, $|\Gamma|$ the length of the curve Γ and C^ε_3 is given in Lemma 2.1.9.

Proof. Let Γ be the curve mentioned above. From Lemma 2.1.7, taking $K_0 = \Gamma$, we have that $\Gamma \subset \rho(-A_\varepsilon)$ for $0 \leq \varepsilon \leq \varepsilon_0(\Gamma)$ with $\varepsilon_0(\Gamma)$ small enough. The spectral projection over the eigenspace generated by the part of the spectrum of $-A_\varepsilon$ contained "inside" the curve Γ is given by

$$\mathbf{P}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\Gamma}} = \frac{1}{2\pi i}\int_\Gamma (A_\varepsilon + \lambda I)^{-1} d\lambda, \qquad \text{with} \quad \lambda \in \Gamma, \quad 0 \leq \varepsilon \leq \varepsilon_0.$$

Therefore,

$$\|\mathbf{P}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\Gamma}}E(v) - E\mathbf{P}^{\mathbf{0}}_{\boldsymbol{\Gamma}}(v)\|_{X^\alpha_\varepsilon} \leq \left|\frac{1}{2\pi i}\right| \left|\int_\Gamma \|(\lambda I + A_\varepsilon)^{-1}E(v) - E(\lambda I + A_0)^{-1}(v)\|_{X^\alpha_\varepsilon} d\lambda\right|.$$

Applying now Lemma 2.1.9, we obtain

$$\|\mathbf{P}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\Gamma}}E(v) - E\mathbf{P}^{\mathbf{0}}_{\boldsymbol{\Gamma}}(v)\|_{X^\alpha_\varepsilon} \leq \frac{1}{2\pi}|\Gamma| \sup_{\lambda\in\Gamma} C^\varepsilon_3(\lambda)\tau(\varepsilon)\|v\|_{X_0} = C_P\tau(\varepsilon)\|v\|_{X_0}. \tag{2.1.25}$$

Since the curve Γ encircles only the first m eigenvalues of $-A_0$, then we know that $\mathbf{P}^{\mathbf{0}}_{\boldsymbol{\Gamma}} = \mathbf{P}^{\mathbf{0}}_{\mathbf{m}}$, that is, the projection over the first m eigenfunctions. This implies that $Rank(\mathbf{P}^{\mathbf{0}}_{\boldsymbol{\Gamma}}) = m$ and from (2.1.25), we also have that $Rank(\mathbf{P}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\Gamma}}) = m$ and therefore we also have $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\Gamma}} = \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}$. Hence, (2.1.25) proves the result. ■

Remark 2.1.13. *With a similar argument as the one in the proof of Lemma 2.1.12, we may prove the continuity of the eigenvalues and of the spectral projections. If λ_0 is an eigenvalue of $-A_0$ of multiplicity s and if $\Gamma = \{z \in \mathbb{C} : |z - \lambda_0| = \delta\}$, with $\delta > 0$ small enough so that $\sigma(-A_0) \cap \{z \in \mathbb{C} : |z - \lambda_0| \leq 2\delta\} = \{\lambda_0\}$, then for ε small enough, $\Gamma \subset \rho(-A_\varepsilon)$ and*

$$\|\mathbf{P}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\Gamma}}E(v) - E\mathbf{P}^{\mathbf{0}}_{\boldsymbol{\Gamma}}(v)\|_{X^\alpha_\varepsilon} \leq C\tau(\varepsilon)\|v\|_{X_0} \to 0, \text{ as } \varepsilon \to 0,$$

which implies that the rank of the projection $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\Gamma}}$ is also s and therefore there are exactly s eigenvalues (counting multiplicity) of $-A_\varepsilon$ in $\{z \in \mathbb{C} : |z - \lambda_0| \leq \delta\}$ and the projections converge.

We can also obtain good estimates for the linear semigroups.

Lemma 2.1.14. *Let hypothesis* **(H1)** *be satisfied. If we denote,*

$$l_\varepsilon^\alpha(t) := \min\{t^{-1}\tau(\varepsilon), t^{-\alpha}\}, \qquad t>0 \quad \textit{and} \quad \alpha\in[0,1)$$

then,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le \max\{4,2\kappa\} l_\varepsilon^\alpha(t). \tag{2.1.26}$$

Proof. Let $\Sigma_{0,\phi}=\{\lambda\in\mathbb{C} : |arg(\lambda)|\le \pi-\phi\}$, with $\phi=\frac{\pi}{4}$, and let Γ be the boundary of $\Sigma_{0,\frac{\pi}{4}}$, that is the curve consisting of the following segments Γ^1 and Γ^2,

$$\Gamma=\Gamma^1\cup\Gamma^2=\{re^{-i(\pi-\phi)} : 0\le r<\infty\}\cup\{re^{i(\pi-\phi)} : 0\le r<+\infty\}$$

oriented such that the imaginary part grows as λ runs in Γ. We know that,

$$e^{-A_\varepsilon t}E - Ee^{-A_0 t} = \frac{1}{2\pi i}\int_\Gamma \left((\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right)e^{\lambda t}d\lambda.$$

So, using Lemma 2.1.9,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le \frac{1}{2\pi}\left|\int_\Gamma C_3\tau(\varepsilon)|e^{\lambda t}|d\lambda\right|,$$

with $C_3=\sup_{\lambda\in\Gamma}C_3^\varepsilon(\lambda)$. Since $\lambda\in\Gamma$,

$$|e^{\lambda t}| = |e^{(re^{-i(\pi-\phi)})t}| = e^{(-rcos(\phi))t} \qquad \text{for} \quad 0\le r\le+\infty, \quad \lambda\in\Gamma^1$$

and,

$$|e^{\lambda t}| = |e^{(re^{i(\pi-\phi)})t}| = e^{(-rcos(\phi))t} \qquad \text{for} \quad 0\le r\le+\infty, \quad \lambda\in\Gamma^2.$$

With this,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le \frac{2}{2\pi}C_3\tau(\varepsilon)\int_0^\infty e^{(-rcos(\phi))t}dr.$$

We make the change of variables $(rcos(\phi))t=z$, and then,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le \frac{1}{\pi}C_3\tau(\varepsilon)\frac{1}{cos(\phi)t}\int_0^\infty e^{-z}dz \le \frac{1}{\pi cos(\phi)}C_3\tau(\varepsilon)t^{-1},$$

with $C_3=\sup_{\lambda\in\Gamma}C_3^\varepsilon(\lambda)\le 6$ and, for $\phi=\frac{\pi}{4}$, $\frac{C_3}{\pi cos(\phi)}<4$, which implies

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le 4\tau(\varepsilon)t^{-1}. \tag{2.1.27}$$

On the other hand,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le \|e^{-A_\varepsilon t}E\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} + \|Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha))}.$$

Then, by Lemma 2.1.6 and (2.1.3),

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le \kappa e^{-\lambda_1^\varepsilon t}\left(\max\{\lambda_1^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha + \kappa e^{-\lambda_1^0 t}\left(\max\{\lambda_1^0, \frac{\alpha}{t}\}\right)^\alpha$$

But, direct computations show that for each $\lambda > 0$ we have $e^{-\lambda t}\left(\max\{\lambda, \frac{\alpha}{t}\}\right)^\alpha \le t^{-\alpha}$ and therefore,

$$\|e^{-A_\varepsilon t}E - Ee^{-A_0 t}\|_{\mathcal{L}(X_0,X_\varepsilon^\alpha)} \le 2\kappa t^{-\alpha} \tag{2.1.28}$$

Putting together (2.1.27) and (2.1.28), we get the result. ∎

For further analysis we will include here some properties of the function $l_\varepsilon^\alpha(t)$ that will be used below.

Lemma 2.1.15. *Let $0 \le \gamma < 1$ and $a > 0$. If we consider, for all $t > 0$,*

$$l_\varepsilon^\alpha(t) := \min\{t^{-1}\tau(\varepsilon), t^{-\alpha}\}, \qquad \text{with} \quad 0 \le \alpha < 1, \quad \text{and} \quad \tau(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0,$$

then, we have the following estimates,

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le \frac{2^\gamma}{(1-\gamma)(1-\alpha)} t^{-\gamma}(|\log(t)) + |\log(\tau(\varepsilon))|)\tau(\varepsilon),$$

$$\int_0^t e^{-as} l_\varepsilon^\alpha(s)ds \le \frac{2}{1-\alpha}(|\log(t)| + |\log(\tau(\varepsilon))|)\tau(\varepsilon),$$

and,

$$\int_0^\infty e^{-as} l_\varepsilon^\alpha(s)ds \le \frac{2}{1-\alpha}|\log(\tau(\varepsilon))|\tau(\varepsilon), \qquad \text{if} \quad a \ge 1.$$

Proof. To prove the first estimate, we divide the analysis in several cases. First, if $0 < t \le 2\tau(\varepsilon)^{\frac{1}{1-\alpha}}$, we have

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le \int_0^t (t-s)^{-\gamma} s^{-\alpha} ds = t^{-\gamma+1-\alpha}\int_0^1 (1-z)^{-\gamma} z^{-\alpha} dz$$

where we have performed the change of variables $s = tz$ in the integral. Hence,

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le Ct^{-\gamma}t^{1-\alpha} \le Ct^{-\gamma}\tau(\varepsilon).$$

Second, if $2\tau(\varepsilon)^{\frac{1}{1-\alpha}} \le t$, then

$$\int_0^t (t-s)^{-\gamma} l_\varepsilon^\alpha(s)ds \le \int_0^{\tau(\varepsilon)^{\frac{1}{1-\alpha}}} (t-s)^{-\gamma} s^{-\alpha} ds + \int_{\tau(\varepsilon)^{\frac{1}{1-\alpha}}}^{t/2} (t-s)^{-\gamma} s^{-1}\tau(\varepsilon) ds$$

$$+\int_{t/2}^{t}(t-s)^{-\gamma}s^{-1}\tau(\varepsilon)ds = I_1+I_2+I_3.$$

We study each term separately. For the first one, I_1, note that if $t\geq 2\tau(\varepsilon)^{\frac{1}{1-\alpha}}$ and $s\in[0,\tau(\varepsilon)^{\frac{1}{1-\alpha}}]$ then $t-s\geq \frac{t}{2}$. So,

$$I_1\leq\left(\frac{t}{2}\right)^{-\gamma}\int_0^{\tau(\varepsilon)^{\frac{1}{1-\alpha}}}s^{-\alpha}ds\leq 2^{\gamma}t^{-\gamma}\frac{1}{1-\alpha}\tau(\varepsilon),$$

$$I_2\leq (t/2)^{-\gamma}(\log(t/2)-\log(\tau(\varepsilon)^{\frac{1}{1-\alpha}}))\tau(\varepsilon)\leq 2^{\gamma}t^{-\gamma}(|\log(t)|+\frac{1}{1-\alpha}|\log(\tau(\varepsilon))|)\tau(\varepsilon),$$

$$I_3\leq t^{-\gamma}\int_{1/2}^{1}(1-z)^{-\gamma}z^{-1}dz\tau(\varepsilon)\leq\frac{2^{\gamma}}{1-\gamma}t^{-\gamma}\tau(\varepsilon)\leq\frac{2^{\gamma}}{1-\gamma}\frac{1}{1-\alpha}t^{-\gamma}\tau(\varepsilon).$$

Putting together the three estimates we show the desired estimate,

$$\int_0^t(t-s)^{-\gamma}l_\varepsilon^\alpha(s)ds\leq\frac{2^{\gamma}}{(1-\gamma)(1-\alpha)}t^{-\gamma}(|\log(t)|+|\log(\tau(\varepsilon))|)\tau(\varepsilon).$$

For the second estimate, we proceed as follows,

$$\int_0^t e^{-as}l_\varepsilon^\alpha(s)ds=\int_0^{\tau(\varepsilon)^{\frac{1}{1-\alpha}}}e^{-as}s^{-\alpha}ds+\int_{\tau(\varepsilon)^{\frac{1}{1-\alpha}}}^{t}e^{-as}s^{-1}\tau(\varepsilon)\leq$$

$$\leq\frac{1}{1-\alpha}\tau(\varepsilon)+e^{-a\tau(\varepsilon)^{\frac{1}{1-\alpha}}}\tau(\varepsilon)\left|\log(t)-\left(\frac{1}{1-\alpha}\right)\log(\tau(\varepsilon))\right|\leq$$

$$\leq\frac{2}{1-\alpha}(|\log(t)|+|\log(\tau(\varepsilon))|)\tau(\varepsilon),$$

as we wanted to prove. For the last one, we write,

$$\int_0^\infty e^{-as}l_\varepsilon^\alpha(s)ds=\int_0^{\tau(\varepsilon)^{\frac{1}{1-\alpha}}}s^{-\alpha}ds+\int_{\tau(\varepsilon)^{\frac{1}{1-\alpha}}}^{1}s^{-1}\tau(\varepsilon)ds+\tau(\varepsilon)\int_1^\infty e^{-as}s^{-1}ds=$$

$$=\frac{\tau(\varepsilon)}{1-\alpha}+\frac{1}{1-\alpha}|\log(\tau(\varepsilon))|\tau(\varepsilon)+\frac{e^{-a}}{a}\tau(\varepsilon)\leq\frac{2e^{-a}}{a(1-\alpha)}|\log(\tau(\varepsilon))|\tau(\varepsilon).$$

Note that, if $a\geq 1$ then,

$$\int_0^\infty e^{-as}l_\varepsilon^\alpha(s)ds\leq\frac{2}{1-\alpha}|\log(\tau(\varepsilon))|\tau(\varepsilon).$$

This concludes the proof of the result. ■

Remark 2.1.16. *If $t=1$, the first estimate is simplified to*

$$\int_0^1(1-s)^{-\gamma}l_\varepsilon^\alpha(s)ds\leq\frac{2^{\gamma}}{(1-\gamma)(1-\alpha)}|\log(\tau(\varepsilon))|\tau(\varepsilon). \tag{2.1.29}$$

2.1.3. Existence of Inertial Manifolds

Our objective in this section is to construct inertial manifolds $\mathcal{M}_0^\varepsilon$, $\mathcal{M}_\varepsilon$, for each $0 < \varepsilon \leq \varepsilon_0$, which will be invariant manifolds for the semi flow generated by (2.1.1) and (2.1.2), therefore proving Proposition 2.1.2. For this purpose, we will use the Lyapunov-Perron method, see [52]. This method consists in constructing the inertial manifold as the graph of a Lipschitz map, which is obtained as the fixed point of an appropriate transformation. For that, observe that Lemma 2.1.7 and Remark 2.1.8 give us that if the operator A_0 has spectral gap, then the operator A_ε will also have it for ε small enough. This spectral gap is essential in the construction of the inertial manifold.

To obtain these inertial manifolds $\mathcal{M}_0^\varepsilon$, $\mathcal{M}_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, consider $m \in \mathbb{N}$ such that $\lambda_m^0 < \lambda_{m+1}^0$ (and therefore $\lambda_m^\varepsilon < \lambda_{m+1}^\varepsilon$ for ε small enough) and denote by $\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}$ the canonical orthogonal projection onto the eigenfunctions, $\{\varphi_i^\varepsilon\}_{i=1}^m$, corresponding to the first m eigenvalues of the operator A_ε, $0 \leq \varepsilon \leq \varepsilon_0$ and $\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}$ the projection over its orthogonal complement, see (3.4.4) and (3.4.5). By technical reasons, we express any element belonging to the linear subspace $\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(X_\varepsilon)$ in the following basis,

$$\{\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_1^0), \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_2^0), ..., \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_m^0)\}, \qquad \text{for} \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

with $\{\varphi_i^0\}_{i=1}^m$ the eigenfunctions related to the first m eigenvalues of A_0. Observe that if $\{\varphi_1^0, \varphi_2^0, ..., \varphi_m^0\}$ is a linearly independent set of vectors, then for ε small enough $\{\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_1^0), \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_2^0), ..., \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_m^0)\}$ is also linearly independent. The proof of this result goes as follows. If we have a linear combination of

$$\{\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_1^0), \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_2^0), ..., \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_m^0)\}$$

such that,

$$\sum_{i=1}^m a_i \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(E\varphi_i^0) = 0,$$

then, from Lemma 2.1.12, choosing $v = \sum_{i=1}^m a_i \varphi_i^0 \in X_0$, we have,

$$\|\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E(v) - E\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}(v)\|_{X_\varepsilon^\alpha} \leq C_P \tau(\varepsilon) \|v\|_{X_0},$$

which implies, since $\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E(v) = 0$,

$$\|E\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}(v)\|_{X_\varepsilon^\alpha} \leq C_P \tau(\varepsilon) \|v\|_{X_0}.$$

But,

$$E\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}(v) = E\mathbf{P}_{\mathbf{m}}^{\mathbf{0}} \left(\sum_{i=1}^m a_i \varphi_i^0 \right) = \sum_{i=1}^m a_i E\varphi_i^0 = Ev.$$

Hence, applying (2.1.5),

$$\|v\|_{X_0} \leq \kappa \|Ev\|_{X_\varepsilon} \leq \kappa \|Ev\|_{X_\varepsilon^\alpha} \leq \kappa C_P \tau(\varepsilon) \|v\|_{X_0},$$

which implies $\|Ev\|_{X^\alpha_\varepsilon} = 0$ if ε is small enough and therefore $v = 0$. But, since $\{\varphi_1^0, \varphi_2^0, ..., \varphi_m^0\}$ is a lineary independent set of vectors, we have $a_i = 0$, $i = 1, ..., m$. Which shows the result.

The Lyapunov-Perron method obtains $\mathcal{M}_0^\varepsilon$, $\mathcal{M}_\varepsilon$ as the graphs of functions $\Psi_0^\varepsilon : \mathbf{P_m^0} X_0^\alpha \to \mathbf{Q_m^0} X_0^\alpha$, $\Psi_\varepsilon : \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha \to \mathbf{Q_m^\varepsilon} X_\varepsilon^\alpha$ which are obtained as fixed points of the functionals

$$(\mathbf{T_0^\varepsilon}\Psi_0^\varepsilon)(\xi) = \int_{-\infty}^0 e^{A_0 \mathbf{Q_m^0} s} \mathbf{Q_m^0} F_0^\varepsilon(p_0^\varepsilon(s) + \Psi_0^\varepsilon(p_0^\varepsilon(s)))ds, \tag{2.1.30}$$

and

$$(\mathbf{T_\varepsilon}\Psi_\varepsilon)(\eta) = \int_{-\infty}^0 e^{A_\varepsilon \mathbf{Q_m^\varepsilon} s} \mathbf{Q_m^\varepsilon} F_\varepsilon(p_\varepsilon(s) + \Psi_\varepsilon(p_\varepsilon(s)))ds, \tag{2.1.31}$$

where $p_0^\varepsilon(s) \in [\varphi_1^0, \dots, \varphi_m^0]$ is the globally defined solution of

$$\begin{cases} p_t = -A_0 p + \mathbf{P_m^0} F_0^\varepsilon(p + \Psi_0^\varepsilon(p(t))) \\ p(0) = \xi \in [\varphi_1^0, \dots, \varphi_m^0] \end{cases} \tag{2.1.32}$$

and $p_\varepsilon(s) \in [\varphi_1^\varepsilon, \dots, \varphi_m^\varepsilon]$ is the globally defined solution of

$$\begin{cases} p_t = -A_\varepsilon p + \mathbf{P_m^\varepsilon} F_\varepsilon(p + \Psi_\varepsilon(p(t))) \\ p(0) = \eta \in [\varphi_1^\varepsilon, \dots, \varphi_m^\varepsilon]. \end{cases} \tag{2.1.33}$$

Following [52] it can be seen that:

Proposition 2.1.17. *Assume hypotheses* **(H1)** *and* **(H2)** *are satisfied. If m is such that*

$$\lambda_{m+1}^0 - \lambda_m^0 \geq 3(\kappa + 2)L_F[(\lambda_{m+1}^0)^\alpha + (\lambda_m^0)^\alpha]$$

$$(\lambda_m^0)^{1-\alpha} \geq 6(\kappa + 2)L_F(1 - \alpha)^{-1}$$

then equations (2.1.1) *and* (2.1.2) *have inertial manifolds $\mathcal{M}_0^\varepsilon$ and $\mathcal{M}_\varepsilon$, respectively, given as the graphs of Lipschitz functions $\Psi_0^\varepsilon : [\varphi_1^0, \dots, \varphi_m^0] \to \mathbf{Q_m^0} X_0^\alpha$ and $\Psi_\varepsilon : [\varphi_1^\varepsilon, \dots, \varphi_m^\varepsilon] \to \mathbf{Q_m^\varepsilon} X_\varepsilon^\alpha$ satisfying,*

$$supp(\Psi_0^\varepsilon) \subset \{\phi \in \mathbf{P_m^0} X_0^\alpha, \|\phi\|_{X_0^\alpha} \leq R\}, \qquad supp(\Psi_\varepsilon) \subset \{\phi \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha, \|\phi\|_{X_\varepsilon^\alpha} \leq R\}$$

$$\|\Psi_0^\varepsilon(p)\|_{X_0^\alpha} \leq L_0, \qquad \|\Psi_\varepsilon(p)\|_{X_\varepsilon^\alpha} \leq L_0$$

$$\|\Psi_0^\varepsilon(p) - \Psi_0^\varepsilon(p')\|_{X_0^\alpha} \leq L_1\|p - p'\|_{X_0^\alpha}, \qquad \|\Psi_\varepsilon(p) - \Psi_\varepsilon(p')\|_{X_\varepsilon^\alpha} \leq L_1\|p - p'\|_{X_\varepsilon^\alpha}$$

for certain L_0, L_1 independent of ε. Moreover, these inertial manifolds are exponentially attracting.

Proof. Observe that if m is such that the gap conditions of the proposition hold, then for ε small enough, see Remark 2.1.13, we have

$$\begin{array}{l} \lambda_{m+1}^{\varepsilon} - \lambda_m^{\varepsilon} \geq 6L_F[(\lambda_{m+1}^{\varepsilon})^{\alpha} + (\lambda_m^{\varepsilon})^{\alpha}] \\ (\lambda_m^{\varepsilon})^{1-\alpha} \geq 12L_F(1-\alpha)^{-1} \end{array} \tag{2.1.34}$$

which are the gap conditions needed in [52] to obtain the inertial manifolds for each ε small enough.

■

With the definition of the isomorphism j_ε, $0 \leq \varepsilon \leq \varepsilon_0$, see (2.1.8), we may define now the inertial manifolds $\Phi_0^{\varepsilon} : \mathbb{R}^m \to \mathbf{Q_m^0} X_0^{\alpha}$, $\Phi_\varepsilon : \mathbb{R}^m \to \mathbf{Q_m^{\varepsilon}} X_\varepsilon^{\alpha}$ as $\Phi_0^{\varepsilon} = \Psi_0^{\varepsilon} \circ j_0^{-1}$ and $\Phi_\varepsilon = \Psi_\varepsilon \circ j_\varepsilon^{-1}$. Notice also that since Ψ_0^{ε} and Ψ_ε are fixed points of $\mathbf{T_0^{\varepsilon}}$, $\mathbf{T_{\varepsilon}}$, then the functions Φ_0^{ε} and Φ_ε satisfy,

$$\Phi_0^{\varepsilon}(z) = \int_{-\infty}^{0} e^{A_0 \mathbf{Q_m^0} s} \mathbf{Q_m^0} F_0^{\varepsilon}\big(p_0^{\varepsilon}(s) + \Phi_0^{\varepsilon}(j_0(p_0^{\varepsilon}(s)))\big) ds, \tag{2.1.35}$$

and

$$\Phi_\varepsilon(z) = \int_{-\infty}^{0} e^{A_\varepsilon \mathbf{Q_m^{\varepsilon}} s} \mathbf{Q_m^{\varepsilon}} F_\varepsilon\big(p_\varepsilon(s) + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s)))\big) ds, \tag{2.1.36}$$

where $p_0^{\varepsilon}(s)$, $p_\varepsilon(s)$ are the solutions of (2.1.32) and (2.1.33) with $p_0^{\varepsilon}(0) = j_0^{-1}(z)$ and $p_\varepsilon(0) = j_\varepsilon^{-1}(z)$. That is, $p_0^{\varepsilon}(s)$ and $p_\varepsilon(s)$ are the solutions of

$$\left\{ \begin{array}{l} p_t = -A_0 p + \mathbf{P_m^0} F_0^{\varepsilon}(p + \Phi_0^{\varepsilon} \circ j_0(p(t))) \\ p(0) = j_0^{-1}(z), \end{array} \right. \tag{2.1.37}$$

and

$$\left\{ \begin{array}{l} p_t = -A_\varepsilon p + \mathbf{P_m^{\varepsilon}} F_\varepsilon(p + \Phi_\varepsilon \circ j_\varepsilon(p(t))) \\ p(0) = j_\varepsilon^{-1}(z), \end{array} \right. \tag{2.1.38}$$

respectively. It is an easy exercise now to show that these functions Φ_0^{ε} and Φ_ε are the inertial manifolds from Proposition 2.1.2.

2.1.4. Rate of convergence of inertial manifolds

Once we have proved the existence of the inertial manifolds $\mathcal{M}_0^{\varepsilon}$ and $\mathcal{M}_\varepsilon$, $\varepsilon \geq 0$ and therefore we have fixed the value of m, we are interested in obtaining the rate of convergence of these inertial manifolds as $\varepsilon \to 0$. To accomplish this, we will need to subtract the integral expressions (2.1.35) and (2.1.36) and make several estimates on these differences. Therefore, we will need first to obtain good estimates on the behavior of the semigroup acting in the spaces $\mathbf{P_m^{\varepsilon}} X_\varepsilon^{\alpha}$ and $\mathbf{Q_m^{\varepsilon}} X_\varepsilon^{\alpha}$.

Since the value of m is fixed and we have the gap condition from Proposition 2.1.17 without loss of generality we will assume that $\lambda_{m+1}^{\varepsilon} - \lambda_m^{\varepsilon} \geq 3$ for all $0 \leq \varepsilon \leq \varepsilon_0$. This allows us to construct the following rectangular curve, encircling the first m eigenvalues:

$$\Gamma = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3 \cup \Gamma^4,$$

where,

$$\Gamma^1 = \{\lambda \in \mathbb{C} : Re(\lambda) = -\lambda_1^0 + 1 \text{ and } |Im(\lambda)| \leq 1\},$$
$$\Gamma^2 = \{\lambda \in \mathbb{C} : -\lambda_m^0 - 1 \leq Re(\lambda) \leq -\lambda_1^0 + 1 \text{ and } Im(\lambda) = 1\},$$
$$\Gamma^3 = \{\lambda \in \mathbb{C} : Re(\lambda) = -\lambda_m^0 - 1 \text{ and } |Im(\lambda)| \leq 1\},$$
$$\Gamma^4 = \{\lambda \in \mathbb{C} : -\lambda_m^0 - 1 \leq Re(\lambda) \leq -\lambda_1^0 + 1 \text{ and } Im(\lambda) = -1\}.$$

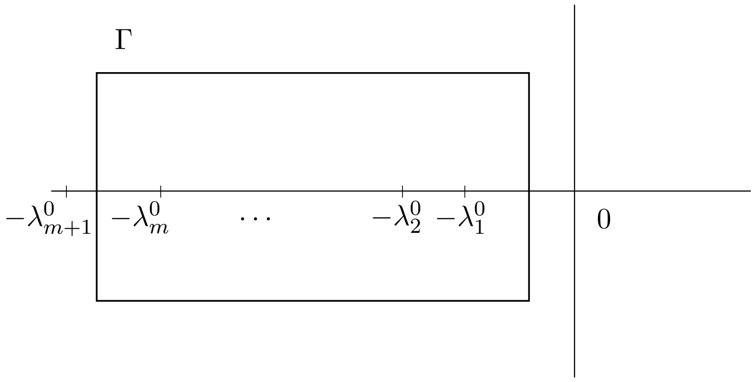

Figure 2.1: Curve Γ

We can prove now,

Lemma 2.1.18. *Let hypothesis* **(H1)** *be satisfied and let* Γ *be the curve defined above. Then,*

$$\|e^{-A_\varepsilon t}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}E - Ee^{-A_0 t}\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq C_4 e^{-(\lambda_m^0+1)t}\tau(\varepsilon), \qquad t \leq 0,$$

with $C_4 = \frac{|\Gamma|}{2\pi}\sup_{\lambda\in\Gamma} C_3^\varepsilon(\lambda)$.

Proof. Since the curve Γ contains the first m eigenvalues of $-A_\varepsilon$, $0 \leq \varepsilon \leq \varepsilon_0$, then

$$e^{-A_\varepsilon t}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}E - Ee^{-A_0 t}\mathbf{P}_{\mathbf{m}}^{\mathbf{0}} = \frac{1}{2\pi i}\int_\Gamma \left((\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\right)e^{\lambda t}d\lambda.$$

So,

$$\|e^{-A_\varepsilon t}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}E - Ee^{-A_0 t}\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}$$
$$\leq \frac{1}{2\pi}\int_\Gamma \|(\lambda I + A_\varepsilon)^{-1}E - E(\lambda I + A_0)^{-1}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}|e^{\lambda t}|d\lambda.$$

Applying Lemma 2.1.9, for $t \leq 0$ we have,

$$\|e^{-A_\varepsilon t}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}E - Ee^{-A_0 t}\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}$$

$$\leq \frac{|\Gamma|}{2\pi} \sup_{\lambda \in \Gamma} C_3^\varepsilon(\lambda) \tau(\varepsilon) \sup_{\lambda \in \Gamma} e^{Re(\lambda)t} = C_4 e^{-(\lambda_m^0+1)t} \tau(\varepsilon),$$

with $C_4 = \frac{|\Gamma|}{2\pi} \sup_{\lambda \in \Gamma} C_3^\varepsilon(\lambda)$ and $|\Gamma|$ the length of the curve Γ. ∎

With respect to the behavior of the linear semigroup in the subspace $\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$, notice that we have the expression

$$e^{-A_\varepsilon t} \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} u = e^{-A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} t} u = \sum_{i=m+1}^{\infty} e^{-\lambda_i^\varepsilon t} (u, \varphi_i^\varepsilon) \varphi_i^\varepsilon.$$

Hence, following a similar proof as Lemma 2.1.6, we get

$$\| e^{-A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} t} \|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq e^{-\lambda_{m+1}^\varepsilon t},$$

and,

$$\| e^{-A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} t} \|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \leq e^{-\lambda_{m+1}^\varepsilon t} \left(\max\{\lambda_{m+1}^\varepsilon, \frac{\alpha}{t}\} \right)^\alpha, \tag{2.1.39}$$

for $t \geq 0$.

In a similar way,

$$e^{-A_\varepsilon t} \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} u = \sum_{i=1}^{m} e^{-\lambda_i^\varepsilon t} (u, \varphi_i^\varepsilon) \varphi_i^\varepsilon.$$

Then, following similar steps as above, for $t \leq 0$ we have,

$$\| e^{-A_\varepsilon \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} t} \|_{\mathcal{L}(X_\varepsilon, X_\varepsilon)} \leq e^{-\lambda_m^\varepsilon t}, \qquad \| e^{-A_\varepsilon \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} t} \|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon^\alpha)} \leq e^{-\lambda_m^\varepsilon t}, \tag{2.1.40}$$

$$\| e^{-A_\varepsilon \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} t} \|_{\mathcal{L}(X_\varepsilon, X_\varepsilon^\alpha)} \leq e^{-\lambda_m^\varepsilon t} (\lambda_m^\varepsilon)^\alpha. \tag{2.1.41}$$

Before continuing, we now present technical lemmas henceforward needed.

Lemma 2.1.19. *Let a be a positive constant, $a > 0$, $\alpha \in (0,1)$ and $\lambda > 0$ a positive real number. We have the following estimate,*

$$\int_0^\infty e^{-as} \left(\max\{\lambda, \frac{\alpha}{s}\} \right)^\alpha ds \leq (1-\alpha)^{-1} \lambda^{\alpha-1} + \lambda^\alpha a^{-1}.$$

Proof. Let $\alpha \in (0,1)$ and λ a real positive number. Then we know that

$$\max\{\lambda, \frac{\alpha}{s}\} = \begin{cases} \frac{\alpha}{s} & \text{if} \quad 0 < s \leq \frac{\alpha}{\lambda} \\ \lambda & \text{if} \quad \frac{\alpha}{\lambda} < s < \infty. \end{cases}$$

So,

$$\int_0^\infty \left(\max\{\lambda, \frac{\alpha}{s}\} \right)^\alpha e^{-as} ds = \int_0^{\frac{\alpha}{\lambda}} \left(\frac{\alpha}{s} \right)^\alpha e^{-as} ds + \int_{\frac{\alpha}{\lambda}}^\infty \lambda^\alpha e^{-as} ds =$$

$$= \alpha^\alpha \int_0^{\frac{\alpha}{\lambda}} s^{-\alpha} e^{-as} ds + \lambda^\alpha \int_{\frac{\alpha}{\lambda}}^{\infty} e^{-as} ds =$$

$$= \alpha^\alpha \left(\frac{\alpha}{\lambda}\right)^{1-\alpha} (1-\alpha)^{-1} + \lambda^\alpha e^{-\frac{a\alpha}{\lambda}} a^{-1} \leq$$

$$\leq (1-\alpha)^{-1} \lambda^{\alpha-1} + \lambda^\alpha a^{-1},$$

as we wanted to prove. ■

Now we want to compare both semigroups $e^{-A_\varepsilon t}$ and $e^{-A_0 t}$ in $\mathbf{Q_m^\varepsilon} X_\varepsilon^\alpha$ and $\mathbf{Q_m^0} X_0^\alpha$. For this, we define first the curve Γ_m which is given by the boundary of $\Sigma_{b,\phi} = \{\lambda \in \mathbb{C} : |arg(\lambda - b)| \leq \pi - \phi\}$, with $\phi = \frac{\pi}{4}$ and $b = -\lambda_{m+1}^0 + 1$. That is,

$$\Gamma_m = \Gamma_m^1 \cup \Gamma_m^2 = \{b + re^{-i(\pi-\phi)} : 0 \leq r < \infty\} \cup \{b + re^{i(\pi-\phi)} : 0 \leq r < +\infty\},$$

oriented such that the imaginary part grows as λ runs in Γ.

We have the following estimates,

Lemma 2.1.20. *Let hypothesis* **(H1)** *be satisfied. If, for $t > 0$, as before we denote by*

$$l_\varepsilon^\alpha(t) := \min\{t^{-1}\tau(\varepsilon), t^{-\alpha}\},$$

then, for each $t > 0$,

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - E e^{-A_0 t} \mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} \leq C_5 e^{-(\lambda_{m+1}^0 - 1)t} l_\varepsilon^\alpha(t),$$

where $C_5 = \max\{\frac{\sup_{\lambda \in \Gamma_m} C_3^\varepsilon(\lambda)}{\pi cos(\phi)}, 2\kappa\}$ and $C_3^\varepsilon(\lambda)$ is defined in Lemma 2.1.9.

Proof. From Lemma 2.1.7 and Remark 2.1.8, we know that there is a real number $\varepsilon_0 = \varepsilon_0(m)$ such that, for $0 \leq \varepsilon \leq \varepsilon_0$, there is a gap between the mth-eigenvalue, $-\lambda_m^\varepsilon$, and the $(m+1)$-eigenvalue, $-\lambda_{m+1}^\varepsilon$, of $-A_\varepsilon$. We denote by Γ_m the boundary of $\Sigma_{b,\phi} = \{\lambda \in \mathbb{C} : |arg(\lambda - b)| \leq \pi - \phi\}$, with $\phi = \frac{\pi}{4}$ and $b = -\lambda_{m+1}^0 + 1$. That is,

$$\Gamma_m = \Gamma_m^1 \cup \Gamma_m^2 = \{b + re^{-i(\pi-\phi)} : 0 \leq r < \infty\} \cup \{b + re^{i(\pi-\phi)} : 0 \leq r < +\infty\},$$

oriented such that the imaginary part grows as λ runs in Γ.

With this,

$$e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - E e^{-A_0 t} \mathbf{Q_m^0} = \frac{1}{2\pi i} \int_{\Gamma_m} \left((\lambda + A_\varepsilon)^{-1} E - E(\lambda + A_0)^{-1}\right) e^{\lambda t} d\lambda.$$

Then,

$$\|e^{-A_\varepsilon t}\mathbf{Q_m^\varepsilon} E - E e^{-A_0 t} \mathbf{Q_m^0}\|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)}$$

$$\leq \frac{1}{2\pi} \left| \int_{\Gamma_m} \| \left((\lambda + A_\varepsilon)^{-1} E - E(\lambda + A_0)^{-1}\right) \|_{\mathcal{L}(X_0, X_\varepsilon^\alpha)} |e^{\lambda t}| d\lambda \right|,$$

applying Lemma 2.1.9

$$\|e^{-A_\varepsilon t}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E - Ee^{-A_0 t}\mathbf{Q}^{\mathbf{0}}_{\mathbf{m}}\|_{\mathcal{L}(X_0,X^\alpha_\varepsilon)}$$

$$\leq \frac{\sup_{\lambda\in\Gamma_m} C^\varepsilon_3(\lambda)\tau(\varepsilon)}{2\pi}\left|\int_{\Gamma_m}|e^{\lambda t}|d\lambda\right| = \frac{\sup_{\lambda\in\Gamma_m} C^\varepsilon_3(\lambda)\tau(\varepsilon)}{\pi}\left|\int_{\Gamma^2_m}|e^{\lambda t}|d\lambda\right|.$$

Since $\lambda \in \Gamma^2_m$,

$$|e^{\lambda t}| = e^{(b-rcos(\phi))t}.$$

So,

$$\|e^{-A_\varepsilon t}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E - Ee^{-A_0 t}\mathbf{Q}^{\mathbf{0}}_{\mathbf{m}}\|_{\mathcal{L}(X_0,X^\alpha_\varepsilon)}$$

$$\leq \frac{\sup_{\lambda\in\Gamma_m} C^\varepsilon_3(\lambda)\tau(\varepsilon)}{\pi}\int_0^\infty e^{(b-rcos(\phi))t}|e^{-i(\pi-\phi)}|dr.$$

We make the change of variables $(-b + rcos(\phi))t = z$,

$$\|e^{-A_\varepsilon t}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E - Ee^{-A_0 t}\mathbf{Q}^{\mathbf{0}}_{\mathbf{m}}\|_{\mathcal{L}(X_0,X^\alpha_\varepsilon)} \leq \frac{\sup_{\lambda\in\Gamma_m} C^\varepsilon_3(\lambda)\tau(\varepsilon)}{\pi cos(\phi)t}\int_{-bt}^\infty e^{-z}dz =$$

$$= \frac{\sup_{\lambda\in\Gamma_m} C^\varepsilon_3(\lambda)}{\pi cos(\phi)}t^{-1}e^{-(\lambda^0_{m+1}-1)t}\tau(\varepsilon).$$

On the other side, we know that,

$$\|e^{-A_\varepsilon t}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E - Ee^{-A_0 t}\mathbf{Q}^{\mathbf{0}}_{\mathbf{m}}\|_{\mathcal{L}(X_0,X^\alpha_\varepsilon)} \leq$$

$$\|e^{-A_\varepsilon t}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E\|_{\mathcal{L}(X_0,X^\alpha_\varepsilon)} + \|Ee^{-A_0 t}\mathbf{Q}^{\mathbf{0}}_{\mathbf{m}}\|_{\mathcal{L}(X_0,X^\alpha_\varepsilon)}.$$

Then, by (2.1.3) and (2.1.39),

$$\leq \kappa e^{-\lambda^\varepsilon_{m+1}t}\left(\max\{\lambda^\varepsilon_{m+1}, \frac{\alpha}{t}\}\right)^\alpha + \kappa e^{-\lambda^0_{m+1}t}\left(\max\{\lambda^0_{m+1}, \frac{\alpha}{t}\}\right)^\alpha \leq$$

$$\leq 2\kappa e^{-(\lambda^0_{m+1}-1)t}\left(\max\{(\lambda^0_{m+1}+1)^\alpha, t^{-\alpha}\}\right).$$

So, if we put everything together,

$$\|e^{-A_\varepsilon t}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E - Ee^{-A_0 t}\mathbf{Q}^{\mathbf{0}}_{\mathbf{m}}\|_{\mathcal{L}(X_0,X^\alpha_\varepsilon)}$$

$$\leq C_5 \min\left\{t^{-1}\tau(\varepsilon), \max\{(\lambda^0_{m+1}+1)^\alpha, t^{-\alpha}\}\right\}e^{-(\lambda^0_{m+1}-1)t} =$$

$$= C_5 \min\left\{t^{-1}\tau(\varepsilon), t^{-\alpha}\right\}e^{-(\lambda^0_{m+1}-1)t} = C_5 l^\alpha_\varepsilon e^{-(\lambda^0_{m+1}-1)t},$$

as we wanted to prove. ■

We may show now the following result.

Lemma 2.1.21. *Let $w_\varepsilon \in \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}X_\varepsilon$ and $w_0 \in \mathbf{P}^{\mathbf{0}}_{\mathbf{m}}X_0$. Then, for ε small enough and for $0 \leq \alpha < 1$,*

$$|j_\varepsilon(w_\varepsilon) - j_0(w_0)|_{0,\alpha} \leq (\kappa+1)\|w_\varepsilon - Ew_0\|_{X^\alpha_\varepsilon} + (\kappa+1)C_P\tau(\varepsilon)\|w_0\|_{X_0},$$

$$|j_\varepsilon(w_\varepsilon) - j_0(w_0)|_{\varepsilon,\alpha} \leq (\kappa+1)\|w_\varepsilon - Ew_0\|_{X^\alpha_\varepsilon} + (\kappa+1)C_P\tau(\varepsilon)\|w_0\|_{X_0},$$

where C_P is the constant from Lemma 2.1.12.

Proof. Recall that the norms $|\cdot|_{0,\alpha}$ and $|\cdot|_{\varepsilon,\alpha}$ are defined in (2.1.10). Since m is fixed and we know that $\lambda_i^\varepsilon \to \lambda_i^0$ for all $i = 1, \dots, m$, then it is not difficult to see that for each $\delta > 0$ we have $\varepsilon(\delta) > 0$ such that

$$(1-\delta)|z|_{0,\alpha} \le |z|_{\varepsilon,\alpha} \le (1+\delta)|z|_{0,\alpha}, \quad 0 \le \varepsilon \le \varepsilon(\delta) \tag{2.1.42}$$

Since $\varphi_i^0 = \mathbf{P_m^0}(\varphi_i^0)$, then if we denote by $j_\varepsilon(w_\varepsilon) = z^\varepsilon$ and $j_0(w_0) = z^0$, we get

$$\begin{aligned} w_\varepsilon - Ew_0 &= \sum_{I=1}^{m} z_i^\varepsilon \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(E\varphi_i^0) - E\sum_{i=1}^{m} z_i^0 \mathbf{P_m^0}\varphi_i^0 \\ &= (\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}E - E\mathbf{P_m^0})\left(\sum_{I=1}^{m} z_i^\varepsilon \varphi_i^0\right) + E\sum_{i=1}^{m}(z_i^\varepsilon - z_i^0)\varphi_i^0 \end{aligned}$$

Applying the operator M and using that $M \circ E = I$, we get

$$\sum_{i=1}^{m}(z_i^\varepsilon - z_i^0)\varphi_i^0 = M(w_\varepsilon - Ew_0) - M(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}E - E\mathbf{P_m^0})\left(\sum_{I=1}^{m} z_i^\varepsilon \varphi_i^0\right)$$

Taking the X_0^α norm in the last expression and with (2.1.3), Lemma 2.1.12 and (2.1.11), we get

$$\begin{aligned} |z^\varepsilon - z^0|_{0,\alpha} &\le \kappa \|w_\varepsilon - Ew_0\|_{X_\varepsilon^\alpha} + \kappa C_p \tau(\varepsilon)|z^\varepsilon| \\ &\le \kappa \|w_\varepsilon - Ew_0\|_{X_\varepsilon^\alpha} + \kappa C_p \tau(\varepsilon)|z^\varepsilon - z^0| + \kappa C_p \tau(\varepsilon)|z^0|. \end{aligned}$$

From here, we get

$$|z^\varepsilon - z^0|_{0,\alpha} \le \frac{\kappa}{1 - \kappa C_P \tau(\varepsilon)}\|w_\varepsilon - Ew_0\|_{X_\varepsilon^\alpha} + \frac{\kappa}{1 - \kappa C_P \tau(\varepsilon)} C_p \tau(\varepsilon)|z^0|. \tag{2.1.43}$$

Taking ε small enough so that $\frac{\kappa}{1-\kappa C_P \tau(\varepsilon)} \le \kappa + 1$ and since $|z^0| = \|w_0\|_{X_0}$, we prove the result for the norm $|\cdot|_{0,\alpha}$.

Now, taking into account (2.1.42) and (2.1.43), for ε small enough we also get the result for the $|\cdot|_{\varepsilon,\alpha}$ norm. ∎

Next, we introduce some technical results.

Lemma 2.1.22. *For every $\Phi_0^\varepsilon \in \mathcal{F}_0(L,R)$, $\Phi_\varepsilon \in \mathcal{F}_\varepsilon(L,R)$, where $\mathcal{F}_0(L,R)$ and $\mathcal{F}_\varepsilon(L,R)$ are defined in* (2.1.12)*, with $L \le 1$ and any $z \in \mathbb{R}^m$, if $p_0^\varepsilon(t)$ is the solution of* (2.1.37) *and $p_\varepsilon(t)$ is the solution of* (2.1.38)*, we have,*

$$\|p_0^\varepsilon(t)\|_{X_0^\alpha} \le \left(\|j_0^{-1}(z)\|_{X_0^\alpha} + \frac{C_F}{(\lambda_m^0)^{1-\alpha}}\right) e^{-\lambda_m^0 t}, \qquad t \le 0,$$

and

$$\|p_\varepsilon(t)\|_{X_\varepsilon^\alpha} \le \left(\|j_\varepsilon^{-1}(z)\|_{X_\varepsilon^\alpha} + \frac{C_F}{(\lambda_m^\varepsilon)^{1-\alpha}}\right) e^{-\lambda_m^\varepsilon t}, \qquad t \le 0.$$

Proof. Both inequalities are obtained in a similar way. Let us prove the second one.

By the variation of constant formula and applying (2.1.40) and (2.1.41) for $t \leq 0$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\|p_\varepsilon(t)\|_{X_\varepsilon^\alpha} \leq \|e^{-A_\varepsilon t} j_\varepsilon^{-1}(z)\|_{X_\varepsilon^\alpha} + \int_t^0 \|e^{-A_\varepsilon(t-s)} \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} F_\varepsilon(p_\varepsilon(s) + \Phi_\varepsilon j_\varepsilon(p_\varepsilon(s)))\|_{X_\varepsilon^\alpha} ds$$

$$\leq e^{-\lambda_m^\varepsilon t} \|j_\varepsilon^{-1}(z)\|_{X_\varepsilon^\alpha} + \int_t^0 e^{-\lambda_m^\varepsilon(t-s)} (\lambda_m^\varepsilon)^\alpha \|\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} F_\varepsilon(p_\varepsilon(s) + \Phi_\varepsilon j_\varepsilon(p_\varepsilon(s)))\|_{X_\varepsilon} ds$$

$$\leq e^{-\lambda_m^\varepsilon t} \|j_\varepsilon^{-1}(z)\|_{X_\varepsilon^\alpha} + \int_t^0 e^{-\lambda_m^\varepsilon(t-s)} (\lambda_m^\varepsilon)^\alpha C_F ds$$

$$\leq \left(\|j_\varepsilon^{-1}(z)\|_{X_\varepsilon^\alpha} + \frac{C_F}{(\lambda_m^\varepsilon)^{1-\alpha}} \right) e^{-\lambda_m^\varepsilon t}.$$

This completes the proof of the lemma. ■

We have now,

Lemma 2.1.23. *With the notations above, we have, for $t \leq 0$,*

$$\|p_\varepsilon(t) - Ep_0^\varepsilon(t)\|_{X_\varepsilon^\alpha} \leq \bigg(\frac{L_F}{(\lambda_m^\varepsilon)^{1-\alpha}} \big(\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} + \rho(\varepsilon) \big)$$
$$+ K_2 e^{-2t} \tau(\varepsilon) \bigg) e^{-(\lambda_m^\varepsilon + (\kappa+2) L_F (\lambda_m^\varepsilon)^\alpha) t}$$

with $K_2 = (2(\kappa+1)(\lambda_m^0)^\alpha L_F C_P + C_4)(|z| + C_F)$ and C_4 is the constant from Lemma 2.1.18.

Proof. To simplify the notation below, we denote by $\tilde{F}_\varepsilon = F_\varepsilon(p_\varepsilon(s) + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s))))$ and similarly, $\tilde{F}_0^\varepsilon = F_0^\varepsilon(p_0^\varepsilon(s) + \Phi_0^\varepsilon(j_0(p_0^\varepsilon(s))))$. By the variation of constants formula applied to (2.1.37) and (2.1.38) we get

$$p_\varepsilon(t) - Ep_0^\varepsilon(t) = e^{-A_\varepsilon t} j_\varepsilon^{-1}(z) - Ee^{-A_0 t} j_0^{-1}(z)$$
$$+ \int_0^t \left(e^{-A_\varepsilon(t-s)} \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} \tilde{F}_\varepsilon - Ee^{-A_0(t-s)} \mathbf{P}_\mathbf{m}^\mathbf{0} \tilde{F}_0^\varepsilon \right) ds$$

$$= e^{-A_\varepsilon t} j_\varepsilon^{-1}(z) - Ee^{-A_0 t} j_0^{-1}(z) + \int_0^t e^{-A_\varepsilon(t-s)} \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} (\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon) ds$$
$$+ \int_0^t (e^{-A_\varepsilon(t-s)} \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E - Ee^{-A_0(t-s)} \mathbf{P}_\mathbf{m}^\mathbf{0}) \tilde{F}_0^\varepsilon ds = I_1 + I_2 + I_3$$

Observe that, with the definition of j_ε, $0 \leq \varepsilon \leq \varepsilon_0$, and with the aid of Lemma 2.1.18, we get

$$\|I_1\|_{X_\varepsilon^\alpha} = \|(e^{-A_\varepsilon t} \mathbf{P}_\mathbf{m}^\varepsilon E - Ee^{-A_0 t} \mathbf{P}_\mathbf{m}^\mathbf{0})(\sum_{i=1}^m z_i \varphi_i^0)\|_{X_\varepsilon^\alpha} \leq C_4 e^{-(\lambda_m^0 + 1)t} \tau(\varepsilon) |z|$$

Moreover, we have

$$\begin{aligned}\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon &= F_\varepsilon(p_\varepsilon + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon))) - F_\varepsilon(Ep_0^\varepsilon + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon)))\\ &+F_\varepsilon(Ep_0^\varepsilon + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon))) - F_\varepsilon(Ep_0^\varepsilon + \Phi_\varepsilon(j_0(p_0^\varepsilon)))\\ &+F_\varepsilon(Ep_0^\varepsilon + \Phi_\varepsilon(j_0(p_0^\varepsilon)) - F_\varepsilon(Ep_0^\varepsilon + E\Phi_0^\varepsilon(j_0(p_0^\varepsilon)))\\ &+F_\varepsilon(Ep_0^\varepsilon + E\Phi_0^\varepsilon(j_0(p_0^\varepsilon)) - EF_0^\varepsilon(p_0^\varepsilon + \Phi_0^\varepsilon(j_0(p_0^\varepsilon)))\end{aligned} \tag{2.1.44}$$

which implies

$$\begin{aligned}\|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon} \leq L_F\|p_\varepsilon - Ep_0^\varepsilon\|_{X_\varepsilon^\alpha} + L_F \cdot L|j_\varepsilon(p_\varepsilon) - j_0(p_0^\varepsilon))|_{\varepsilon,\alpha}\\ +L_F\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)\end{aligned}$$

where we have denoted $\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty = \|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)}$. Taking into account Lemma 2.1.21, we get

$$\begin{aligned}\|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon} \leq (\kappa+2)L_F\|p_\varepsilon - Ep_0^\varepsilon\|_{X_\varepsilon^\alpha} + (\kappa+1)L_F C_P \tau(\varepsilon)\|p_0^\varepsilon\|_{X_0}\\ +L_F\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)\end{aligned}$$

which implies with Lemma 2.1.22, using that $\lambda_m^\varepsilon \geq 1$ and that $\|j_0^{-1}(z)\|_{X_0} = |z|$,

$$\begin{aligned}\|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon} \leq (\kappa+2)L_F\|p_\varepsilon - Ep_0^\varepsilon\|_{X_\varepsilon^\alpha} + (\kappa+1)L_F C_P \tau(\varepsilon)(|z| + C_F)e^{-\lambda_m^0 s}+\\ +L_F\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)\end{aligned} \tag{2.1.45}$$

In particular, we obtain:

$$\|I_2\|_{X_\varepsilon^\alpha} \leq (\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)}\|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon} ds$$

That is,

$$\begin{aligned}\|I_2\|_{X_\varepsilon^\alpha} \leq\ & (\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)}\|p_\varepsilon(s) - Ep_0^\varepsilon(s)\|_{X_\varepsilon^\alpha} ds\\ &+(\lambda_m^\varepsilon)^\alpha(\kappa+1)L_F C_P(|z| + C_F)\tau(\varepsilon)e^{-\lambda_m^\varepsilon t}\int_t^0 e^{(\lambda_m^\varepsilon - \lambda_m^0)s} ds+\\ &+(\lambda_m^\varepsilon)^\alpha\left(L_F\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)\right)\int_t^0 e^{-\lambda_m^\varepsilon(t-s)} ds\\ &\leq (\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)}\|p_\varepsilon(s) - Ep_0^\varepsilon(s)\|_{X_\varepsilon^\alpha} ds\\ &+\left(K_1\tau(\varepsilon)e^{-t} + \frac{L_F}{(\lambda_m^\varepsilon)^{1-\alpha}}(\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon))\right)e^{-\lambda_m^\varepsilon t}\end{aligned}$$

where we have denoted by $K_1 = 2(\kappa+1)(\lambda_m^0)^\alpha L_F C_P(|z|+C_F)$ and we have used that $\lambda_m^\varepsilon > 1$ and that ε is small enough so that $(\lambda_m^\varepsilon)^\alpha \leq 2(\lambda_m^0)^\alpha$ and $|\lambda_m^\varepsilon - \lambda_m^0| < 1$.

Finally,

$$\|I_3\|_{X_\varepsilon^\alpha} \leq C_4\tau(\varepsilon)C_F \int_t^0 e^{-(\lambda_m^0+1)(t-s)}ds \leq C_4\tau(\varepsilon)C_F e^{-(\lambda_m^0+1)t}$$

Putting the three expressions together, we get

$$\begin{aligned}&\|p_\varepsilon(t) - Ep_0^\varepsilon(t)\|_{X_\varepsilon^\alpha} \leq C_4(|z|+C_F)e^{-(\lambda_m^0+1)t}\tau(\varepsilon)+\\ &\left(K_1\tau(\varepsilon)e^{-t} + \frac{L_F}{(\lambda_m^\varepsilon)^{1-\alpha}}(\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon))\right)e^{-\lambda_m^\varepsilon t}\\ &+(\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)}\|p_\varepsilon(s) - Ep_0^\varepsilon(s)\|_{X_\varepsilon^\alpha}ds.\end{aligned}$$

Multiplying this inequality by $e^{\lambda_m^\varepsilon t}$, denoting by $h(t) = e^{\lambda_m^\varepsilon t}\|p_\varepsilon(t) - Ep_0^\varepsilon(t)\|_{X_\varepsilon^\alpha}$, we may write

$$\begin{aligned}h(t) \leq &\left(\frac{L_F}{(\lambda_m^\varepsilon)^{1-\alpha}}(\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)) + K_2e^{-2t}\tau(\varepsilon)\right)\\ &+(\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 h(s)ds\end{aligned}$$

where $K_2 = K_1 + C_4(|z|+C_F)$. Applying Gronwall inequality, we get,

$$h(t) \leq \left(\frac{L_F}{(\lambda_m^\varepsilon)^{1-\alpha}}(\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)) + K_2e^{-2t}\tau(\varepsilon)\right)e^{-(\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha t}$$

which implies that

$$\begin{aligned}\|p_\varepsilon(t) - Ep_0^\varepsilon(t)\|_{X_\varepsilon^\alpha} \leq &\left(\frac{L_F}{(\lambda_m^\varepsilon)^{1-\alpha}}(\sup_{\bar p\in\mathbb{R}^m}\|\Phi_\varepsilon(\bar p) - E\Phi_0^\varepsilon(\bar p)\|_{X_\varepsilon^\alpha} + \rho(\varepsilon))\right.\\ &\left.+K_2e^{-2t}\tau(\varepsilon)\right)e^{-(\lambda_m^\varepsilon+(\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha)t}\end{aligned}$$

which shows the result. ∎

We have the following corollary

Corolary 2.1.24. *We obtain the following estimate for $t = 1$,*

$$\|p_\varepsilon(1) - Ep_0^\varepsilon(1)\|_{X_\varepsilon^\alpha} \leq C\left(\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} + \rho(\varepsilon) + \tau(\varepsilon)\right),$$

with C a constant independent of ε.

Proof. We apply the same notation used in the proof of previous lemma. Then, by the variation of constants formula applied to (2.1.37) and (2.1.38) we get

$$p_\varepsilon(1) - Ep_0^\varepsilon(1) = e^{-A_\varepsilon} j_\varepsilon^{-1}(z) - Ee^{-A_0} j_0^{-1}(z)$$
$$+ \int_0^1 \left(e^{-A_\varepsilon(1-s)} \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} \tilde{F}_\varepsilon - Ee^{-A_0(1-s)} \mathbf{P}^{\mathbf{0}}_{\mathbf{m}} \tilde{F}_0^\varepsilon \right) ds$$
$$= e^{-A_\varepsilon} j_\varepsilon^{-1}(z) - Ee^{-A_0} j_0^{-1}(z) + \int_0^1 e^{-A_\varepsilon(1-s)} \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} (\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon) ds$$
$$+ \int_0^1 (e^{-A_\varepsilon(1-s)} \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} E - Ee^{-A_0(1-s)} \mathbf{P}^{\mathbf{0}}_{\mathbf{m}}) \tilde{F}_0^\varepsilon ds = I_1 + I_2 + I_3.$$

With the aid of Lemma 2.1.18, considering $t = 1$, we obtain,

$$\|I_1\|_{X_\varepsilon^\alpha} = \|(e^{-A_\varepsilon} \mathbf{P}^\varepsilon_{\mathbf{m}} E - Ee^{-A_0} \mathbf{P}^{\mathbf{0}}_{\mathbf{m}})(\sum_{i=1}^m z_i \varphi_i^0)\|_{X_\varepsilon^\alpha} \leq C_4 e^{(-\lambda_1^0+1)} \tau(\varepsilon)|z| = C\tau(\varepsilon).$$

Following the same arguments as in the previous lemma,

$$\|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon} \leq (\kappa+2)L_F\|p_\varepsilon - Ep_0^\varepsilon\|_{X_\varepsilon^\alpha} + (\kappa+1)L_F C_P \tau(\varepsilon)\|p_0^\varepsilon\|_{X_0}$$
$$+ L_F\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)$$

which implies with Lemma 2.1.22 for $t = 1$ and that $\|j_0^{-1}(z)\|_{X_0} = |z|$,

$$\begin{aligned} \|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon} &\leq (\kappa+2)L_F\|p_\varepsilon - Ep_0^\varepsilon\|_{X_\varepsilon^\alpha} + (\kappa+1)L_F C_P \tau(\varepsilon) C + \\ &+ L_F\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon), \end{aligned} \tag{2.1.46}$$

we have denote by the general letter C the bounded of $\|p_0^\varepsilon(1)\|_{X_0}$ which not depends on ε, see the proof of Lemma 2.1.22. Then,

$$\|I_2\|_{X_\varepsilon^\alpha} \leq (\lambda_1^\varepsilon)^\alpha \int_0^1 e^{-\lambda_1^\varepsilon(1-s)} \|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon} ds \leq$$
$$(\kappa+2)L_F(\lambda_1^\varepsilon)^\alpha \int_0^1 e^{-\lambda_1^\varepsilon(1-s)} \|p_\varepsilon(s) - Ep_0^\varepsilon(s)\|_{X_\varepsilon^\alpha} ds +$$
$$+ C\left(\tau(\varepsilon) + \|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)\right)).$$

Finally,

$$\|I_3\|_{X_\varepsilon^\alpha} \leq C_4 \tau(\varepsilon) C_F \int_0^1 e^{(-\lambda_1^0+1)(1-s)} ds = C\tau(\varepsilon)$$

Hence,

$$\|p_\varepsilon(1) - Ep_0^\varepsilon(1)\|_{X_\varepsilon^\alpha} \leq C\left(\tau(\varepsilon) + \|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \rho(\varepsilon)\right)) +$$
$$(\kappa+2)L_F(\lambda_1^\varepsilon)^\alpha \int_0^1 e^{-\lambda_1^\varepsilon(1-s)} \|p_\varepsilon(s) - Ep_0^\varepsilon(s)\|_{X_\varepsilon^\alpha} ds.$$

Applying Gronwall inequality,

$$\|p_\varepsilon(1) - Ep_0^\varepsilon(1)\|_{X_\varepsilon^\alpha} \leq C\left(\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty + \tau(\varepsilon) + \rho(\varepsilon)\right),$$

with C a constant independent of ε which shows the result. ■

With these results, we have all the needed tools to estimate the rate of convergence of the inertial manifolds, proving one of the main results of this chapter.

Proof of Theorem 2.1.4. Notice that we have

$$\Phi_0^\varepsilon(z) = \int_{-\infty}^0 e^{A_0 s}\mathbf{Q}_\mathbf{m}^\mathbf{0} F_0^\varepsilon\big(p_0^\varepsilon(s) + \Phi_0^\varepsilon(j_0(p_0^\varepsilon(s)))\big)ds, \tag{2.1.47}$$

and

$$\Phi_\varepsilon(z) = \int_{-\infty}^0 e^{A_\varepsilon s}\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} F_\varepsilon\big(p_\varepsilon(s) + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s)))\big)ds, \tag{2.1.48}$$

where $p_0^\varepsilon(s)$ and $p_\varepsilon(s)$ are the solutions of (2.1.37) and (2.1.38). Denoting, as in the proof of the previous Lemma, $\tilde{F}_\varepsilon = F_\varepsilon(p_\varepsilon(s) + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s))))$ and $\tilde{F}_0^\varepsilon = F_0^\varepsilon(p_0^\varepsilon(s) + \Phi_0^\varepsilon(j_0(p_0^\varepsilon(s))))$

$$\Phi_\varepsilon(z) - E\Phi_0^\varepsilon(z) = \int_{-\infty}^0 \left(e^{A_\varepsilon s}\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}}\tilde{F}_\varepsilon - Ee^{A_0 s}\mathbf{Q}_\mathbf{m}^\mathbf{0}\tilde{F}_0^\varepsilon\right)ds =$$

$$= \int_{-\infty}^0 e^{A_\varepsilon s}\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}}(\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon)ds + \int_{-\infty}^0 \left(e^{A_\varepsilon s}\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}}E - Ee^{A_0 s}\mathbf{Q}_\mathbf{m}^\mathbf{0}\right)\tilde{F}_0^\varepsilon ds = I_1 + I_2.$$

With (2.1.39)

$$\|I_1\|_{X_\varepsilon^\alpha} \leq \int_{-\infty}^0 e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon, \frac{\alpha}{t}\}\right)^\alpha \|\tilde{F}_\varepsilon - E\tilde{F}_0^\varepsilon\|_{X_\varepsilon}ds.$$

Now, with the decomposition as in (2.1.44) and with (2.1.46) and denoting by $\|E\Phi_0^\varepsilon - \Phi_\varepsilon\|_\infty = \|E\Phi_0^\varepsilon - \Phi_\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)}$, we obtain

$$\|I_1\|_{X_\varepsilon^\alpha} \leq \int_{-\infty}^0 e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon, \frac{\alpha}{s}\}\right)^\alpha \Big[(\kappa+2)L_F\|p_\varepsilon(s) - Ep_0^\varepsilon(s)\|_{X_\varepsilon^\alpha}$$

$$+(\kappa+1)L_F C_P\tau(\varepsilon)(|z| + C_F)e^{-\lambda_m^0 s} + L_F\|E\Phi_0^\varepsilon - \Phi_\varepsilon\|_\infty + \rho(\varepsilon)\Big]ds$$

$$= (\kappa+2)L_F\int_{-\infty}^0 e^{\lambda_{m+1}^\varepsilon s}\left(\max\{\lambda_{m+1}^\varepsilon, \frac{\alpha}{s}\}\right)^\alpha \|p_\varepsilon(s) - Ep_0^\varepsilon(s)\|_{X_\varepsilon^\alpha}ds+$$

$$+(\kappa+1)L_F C_P\tau(\varepsilon)(|z|+C_F)\int_{-\infty}^{0} e^{(\lambda_{m+1}^{\varepsilon}-\lambda_m^0)s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} ds+$$

$$+\rho(\varepsilon)\int_{-\infty}^{0} e^{\lambda_{m+1}^{\varepsilon}s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} ds$$

$$+L_F\|E\Phi_0^{\varepsilon}-\Phi_{\varepsilon}\|_{\infty}\int_{-\infty}^{0} e^{\lambda_{m+1}^{\varepsilon}s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} ds.$$

The second term in the last expression can be estimated with Lemma 2.1.19, since

$$\int_{-\infty}^{0} e^{(\lambda_{m+1}^{\varepsilon}-\lambda_m^0)s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} \le (1-\alpha)^{-1}(\lambda_{m+1}^{\varepsilon})^{\alpha-1}+(\lambda_{m+1}^{\varepsilon})^{\alpha}(\lambda_{m+1}^{\varepsilon}-\lambda_m^0)^{-1}$$

which is uniformly bounded as $\varepsilon\to 0$. Then, the second term is bounded by $C(|z|+1)\tau(\varepsilon)$ with C a constant independent of ε. Similar estimate is obtained for the third term: it will be bounded by $C\rho(\varepsilon)$ with C a constant independent of the parameter ε.

For the fourth term

$$\int_{-\infty}^{0} e^{\lambda_{m+1}^{\varepsilon}s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} \le$$

$$(1-\alpha)^{-1}(\lambda_{m+1}^{\varepsilon})^{\alpha-1}+(\lambda_{m+1}^{\varepsilon})^{\alpha-1}\le 2(1-\alpha)^{-1}(\lambda_{m+1}^{\varepsilon})^{\alpha-1}.$$

Which implies that ,

$$L_F\|E\Phi_0^{\varepsilon}-\Phi_{\varepsilon}\|_{\infty}\int_{-\infty}^{0} e^{\lambda_{m+1}^{\varepsilon}s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} ds\le$$

$$2L_F(1-\alpha)^{-1}(\lambda_{m+1}^{\varepsilon})^{\alpha-1}\|E\Phi_0^{\varepsilon}-\Phi_{\varepsilon}\|_{\infty}$$

The first term need to be estimated with the aid of Lemma 2.1.23. Notice that,

$$(\kappa+2)L_F\int_{-\infty}^{0} e^{\lambda_{m+1}^{\varepsilon}s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha}\|p_{\varepsilon}(s)-Ep_0^{\varepsilon}(s)\|_{X_{\varepsilon}^{\alpha}}ds\le$$

$$\le\frac{(\kappa+2)L_F^2}{(\lambda_m^{\varepsilon})^{1-\alpha}}\|E\Phi_0^{\varepsilon}-\Phi_{\varepsilon}\|_{\infty}\int_{-\infty}^{0} e^{(\lambda_{m+1}^{\varepsilon}-\lambda_m^{\varepsilon}-(\kappa+2)L_F(\lambda_m^{\varepsilon})^{\alpha})s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} ds+$$

$$+(\kappa+2)L_F\rho(\varepsilon)\int_{-\infty}^{0} e^{(\lambda_{m+1}^{\varepsilon}-\lambda_m^{\varepsilon}-(\kappa+2)L_F(\lambda_m^{\varepsilon})^{\alpha})s}\left(\max\{\lambda_{m+1}^{\varepsilon},\frac{\alpha}{s}\}\right)^{\alpha} ds+$$

$$+(\kappa+2)K_2L_F\tau(\varepsilon)\int_{-\infty}^{0}e^{(\lambda^{\varepsilon}_{m+1}-\lambda^{\varepsilon}_{m}-(\kappa+2)L_F(\lambda^{\varepsilon}_{m})^{\alpha})s}\left(\max\{\lambda^{\varepsilon}_{m+1},\frac{\alpha}{s}\}\right)^{\alpha}e^{-2s}ds$$

With similar arguments as above, the last two terms are bounded by $C\rho(\varepsilon)$ and $C\tau(\varepsilon)$ with C a constant independent of ε.

The first term is bounded by

$$\frac{(\kappa+2)L_F^2}{(\lambda^{\varepsilon}_{m})^{1-\alpha}}\|E\Phi^{\varepsilon}_0-\Phi_{\varepsilon}\|_{\infty}\left((1-\alpha)^{-1}(\lambda^{\varepsilon}_{m+1})^{\alpha-1}+\frac{(\lambda^{\varepsilon}_{m+1})^{\alpha}}{\lambda^{\varepsilon}_{m+1}-\lambda^{\varepsilon}_{m}-(\kappa+2)L_F(\lambda^{\varepsilon}_{m})^{\alpha}}\right)$$

Putting all these estimates together, we have

$$\|I_1\|_{X^{\alpha}_{\varepsilon}}\leq\Big[2L_F(1-\alpha)^{-1}(\lambda^{\varepsilon}_{m+1})^{\alpha-1}$$

$$+\frac{(\kappa+2)L_F^2}{(\lambda^{\varepsilon}_{m})^{1-\alpha}}\left((1-\alpha)^{-1}(\lambda^{\varepsilon}_{m+1})^{\alpha-1}+\frac{(\lambda^{\varepsilon}_{m+1})^{\alpha}}{\lambda^{\varepsilon}_{m+1}-\lambda^{\varepsilon}_{m}-(\kappa+2)L_F(\lambda^{\varepsilon}_{m})^{\alpha}}\right)\Big]\|E\Phi^{\varepsilon}_0-\Phi_{\varepsilon}\|_{\infty}+$$

$$+C(|z|+1)\tau(\varepsilon)+C\rho(\varepsilon).$$

and using the gap conditions from Proposition 2.1.17, which in particular imply that for ε small enough we have

$$\begin{array}{l}\lambda^{\varepsilon}_{m+1}-\lambda^{\varepsilon}_{m}\geq\frac{3}{2}(\kappa+2)L_F[(\lambda^{\varepsilon}_{m+1})^{\alpha}+(\lambda^{\varepsilon}_{m})^{\alpha}]\\(\lambda^{\varepsilon}_{m})^{1-\alpha}\geq3(\kappa+2)L_F(1-\alpha)^{-1}\end{array}$$

we easily get

$$\|I_1\|_{X^{\alpha}_{\varepsilon}}\leq\frac{1}{2}\|E\Phi^{\varepsilon}_0-\Phi_{\varepsilon}\|_{\infty}+C(|z|+1)\tau(\varepsilon)+C\rho(\varepsilon).$$

Now we estimate I_2.

$$\begin{aligned}\|I_2\|_{X^{\alpha}_{\varepsilon}}&\leq\int_{-\infty}^{0}\|\left(e^{A_{\varepsilon}s}\mathbf{Q^{\varepsilon}_m}-Ee^{A_0s}\mathbf{Q^0_m}\right)\|_{\mathcal{L}(X_0,X^{\alpha}_{\varepsilon})}\|\tilde{F}^{\varepsilon}_0\|_{X_0}ds\\&\leq\int_{-\infty}^{0}C_5e^{-(\lambda^0_{m+1}-1)t}l^{\alpha}_{\varepsilon}(t)C_Fdt\leq\frac{2C_5C_F}{1-\alpha}\tau(\varepsilon)|\log(\tau(\varepsilon))|\end{aligned}$$

where we have used Lemma 2.1.20 and Lemma 2.1.15.

Putting together the estimates for I_1 and I_2, we get

$$\begin{aligned}\|\Phi_{\varepsilon}(z)-E\Phi^{\varepsilon}_0(z)\|_{X^{\alpha}_{\varepsilon}}&\leq\frac{1}{2}\|\Phi_{\varepsilon}-E\Phi^{\varepsilon}_0\|_{\infty}+C(|z|+1)\tau(\varepsilon)\\&+C\rho(\varepsilon)+\frac{2C_5C_F}{1-\alpha}\tau(\varepsilon)|\log(\tau(\varepsilon))|\end{aligned}$$

Now since Φ_ε and Φ_0^ε are of compact support, we take the sup norm for z with $|z| \leq R$, where R is an upper bound of the support of all inertial manifolds and obtain

$$\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty \leq \frac{1}{2}\|E\Phi_0^\varepsilon - \Phi_\varepsilon\|_\infty + C(R+1)\tau(\varepsilon) + C\rho(\varepsilon) + \frac{2C_5C_F}{1-\alpha}\tau(\varepsilon)|\log(\tau(\varepsilon))|$$

which implies that

$$\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_\infty \leq C(\rho(\varepsilon) + \tau(\varepsilon)|\log(\tau(\varepsilon))|)$$

which shows the theorem. ■

2.2. Smoothness and $C^{1,\theta}$-convergence of inertial manifolds.

In this section we study the smoothness of the inertial manifolds, Φ_0^ε, Φ_ε, $0 < \varepsilon \leq \varepsilon_0$, obtained in Section 2.1 and show that for a fixed ε, the inertial manifolds have a $C^{1,\theta}$ regularity for some appropriate $0 < \theta \leq 1$. We also show that Φ_ε and Φ_0^ε are close not only in the C^0 topology , as Theorem 2.1.4 asserts, but also in the $C^{1,\theta}$ topology. Moreover, we will be able to obtain a rate of its distance in this topology.

2.2.1. Setting and main results

To obtain these results we have to assume some extra hypotheses about the nonlinearities F_0^ε and F_ε, $0 < \varepsilon \leq \varepsilon_0$.

(H2'). *We assume that the nonlinear terms F_0^ε, F_ε, satisfy hypothesis* **(H2)** *and they are uniformly C^{1,θ_F} functions from X_0^α to X_0 and from X_ε^α to X_ε, respectively, for some $0 < \theta_F \leq 1$. That is, $F_0^\varepsilon \in C^1(X_0^\alpha, X_0)$, $F_\varepsilon \in C^1(X_\varepsilon^\alpha, X_\varepsilon)$ and there exists $L > 0$, independent of ε, such that*

$$\|DF_0^\varepsilon(u) - DF_0^\varepsilon(u')\|_{\mathcal{L}(X_0^\alpha, X_0)} \leq L\|u - u'\|_{X_0^\alpha}^{\theta_F}, \qquad \forall u, u' \in X_0^\alpha$$

$$\|DF_\varepsilon(u) - DF_\varepsilon(u')\|_{\mathcal{L}(X_\varepsilon^\alpha, X_\varepsilon)} \leq L\|u - u'\|_{X_\varepsilon^\alpha}^{\theta_F}, \qquad \forall u, u' \in X_\varepsilon^\alpha.$$

We can state now the main results of this section.

Proposition 2.2.1. *Assume hypotheses* **(H1)** *and* **(H2')** *are satisfied and that the gap conditions* (2.1.13), (2.1.14) *hold. Then, for any* $\theta > 0$ *such that* $\theta \leq \theta_F$ *and* $\theta < \theta_0$, *where*

$$\theta_0 = \frac{\lambda^0_{m+1} - \lambda^0_m - 4L_F(\lambda^0_m)^\alpha - 2L_F(\lambda^0_{m+1})^\alpha}{2L_F(\lambda^0_m)^\alpha + \lambda^0_m} \tag{2.2.1}$$

then, the functions Φ^ε_0, Φ_ε, $0 < \varepsilon \leq \varepsilon_0$, *obtained in Section 2.1 which give the inertial manifolds, are* $C^{1,\theta}(\mathbb{R}^m, X^\alpha_0)$ *and* $C^{1,\theta}(\mathbb{R}^m, X^\alpha_\varepsilon)$, *respectively.*

Theorem 2.2.2. *Let hypotheses* **(H1)**, **(H2')** *and gap conditions* (2.1.13), (2.1.14) *be satisfied, so that Proposition 2.2.1 hold, and we have inertial manifolds* $\mathcal{M}^\varepsilon_0$ *and* $\mathcal{M}^\varepsilon$ *given as the graphs of the functions* Φ^ε_0 *and* Φ_ε. *If we denote by*

$$\beta(\varepsilon) = \sup_{u \in \mathcal{M}^\varepsilon_0} \|DF_\varepsilon\big(Eu\big)E - EDF^\varepsilon_0\big(u\big)\|_{\mathcal{L}(X^\alpha_0, X_\varepsilon)}, \tag{2.2.2}$$

then, there exists a $\theta^* < \theta_F$ *such that for all* $0 < \theta < \theta^*$, *we obtain the following estimate*

$$\|E\Phi^\varepsilon_0 - \Phi_\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m, X^\alpha_\varepsilon)} \leq \mathbf{C}\left(\left[\beta(\varepsilon) + \Big(\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)\Big)^{\theta^*}\right]\right)^{1-\frac{\theta}{\theta^*}}, \tag{2.2.3}$$

where $\mathbf{C}$ *is a constant independent of* ε *and* $\tau(\varepsilon)$, $\rho(\varepsilon)$ *are given by (2.1.7), (2.1.15), respectively.*

Remark 2.2.3. *As a matter of fact,* θ^* *can be chosen* $\theta^* < \min\{\theta_F, \theta_0, \theta_1\}$ *where* θ_F *is from* **(H2')**, θ_0 *is defined in*(2.2.1) *and* θ_1,

$$\theta_1 = \frac{\lambda^0_{m+1} - \lambda^0_m - 4L_F(\lambda^0_m)^\alpha}{(\kappa+2)L_F(\lambda^0_m)^\alpha + \lambda^0_m + 3},$$

see (2.2.12).

Throughout this section the space $C^{1,\theta}(\mathbb{R}^m, X^\alpha_\varepsilon)$ is the usual space of $C^1(\mathbb{R}^m, X^\alpha_\varepsilon)$ maps whose differentials are uniformly Hölder continuous with Hölder exponent θ. That is, there is a constant C independent of ε such that,

$$\|D\Phi_\varepsilon(z) - D\Phi_\varepsilon(z')\|_{\mathcal{L}(\mathbb{R}^m, X^\alpha_\varepsilon)} \leq C|z - z'|^\theta_{\varepsilon,\alpha}.$$

where the norm $|\cdot|_{\varepsilon,\alpha}$ is given by (2.1.10). Notice that the norm $|\cdot|_{\varepsilon,\alpha}$ is equivalent to $|\cdot|$ uniformly in ε and α.

The space $C^{1,\theta}(\mathbb{R}^m, X^\alpha_\varepsilon)$ is endowed with the norm $\|\cdot\|_{C^{1,\theta}(\mathbb{R}^m, X^\alpha_\varepsilon)}$ given by,

$$\|\Phi_\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m, X^\alpha_\varepsilon)} = \|\Phi_\varepsilon\|_{C^1(\mathbb{R}^m, X^\alpha_\varepsilon)} + \sup_{z,z' \in \mathbb{R}^m} \frac{\|D\Phi_\varepsilon(z) - D\Phi_\varepsilon(z')\|_{\mathcal{L}(\mathbb{R}^m, X^\alpha_\varepsilon)}}{|z - z'|^\theta_{\varepsilon,\alpha}}$$

To simplify notation below and unless some clarification is needed, we will denote the norms $\|\cdot\|_{C^1(\mathbb{R}^m, X_\varepsilon^\alpha)}$ and $\|\cdot\|_{C^{1,\theta}(\mathbb{R}^m, X_\varepsilon^\alpha)}$ by $\|\cdot\|_{C^1}$ and $\|\cdot\|_{C^{1,\theta}}$. Also, very often we will need to consider the following space of bounded linear operators $\mathcal{L}(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha, \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha)$ and its norm will be abbreviated by $\|\cdot\|_{\mathcal{L}}$.

Remember that these functions Φ_0^ε, Φ_ε, $\varepsilon > 0$, are defined by $\Phi_0^\varepsilon := \Psi_0^\varepsilon \circ j_0^{-1}$, $\Phi_\varepsilon := \Psi_\varepsilon \circ j_\varepsilon^{-1}$ with $\Psi_0^\varepsilon : \mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha \to \mathbf{Q}_\mathbf{m}^\mathbf{0} X_0^\alpha$ and $\Psi_\varepsilon : \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha \to \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$, the fixed points of the functionals $\mathbf{T}_\mathbf{0}^\varepsilon$, $\mathbf{T}_{\boldsymbol{\varepsilon}}$ defined in (2.1.30), (2.1.31) (see also [52]), and j_ε, $0 \leq \varepsilon \leq \varepsilon_0$, the isomorphism which identifies $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$ with $\mathbb{R}^m$, see (2.1.8).

We divide this section in two subsections. In the first one, we show the $C^{1,\theta}$ smoothness of the inertial manifold Φ_0^ε, Φ_ε for a fixed value of the parameter ε. Moreover, we will obtain estimates of its norm in the $C^{1,\theta}$ norm which do not depend on ε. The second subsection is devoted to prove the convergence of the inertial manifolds in the $C^{1,\theta}$ topology, proving Theorem 2.2.2.

2.2.2. Smoothness of inertial manifolds

We analyze in this subsection the smoothness of the inertial manifolds Φ_0^ε, Φ_ε, for a fixed value of the parameter $\varepsilon \in [0, \varepsilon_0]$. Recall that the C^1 smoothness of the manifold is shown in [52], where they proved the following result:

Theorem 2.2.4. *Let hypotheses of Proposition 2.1.2 be satisfied. Assume that for each $\varepsilon \geq 0$ the nonlinear functions F_0^ε, F_ε are Lipschitz C^1 functions from X_0^α to X_0 and from X_ε^α to X_ε. Then, the inertial manifolds $\mathcal{M}_0^\varepsilon$, $\mathcal{M}_\varepsilon$, $\varepsilon > 0$, are C^1-manifolds and the functions Ψ_0^ε, Ψ_ε are Lipschitz C^1 functions from $\mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha$ to $\mathbf{Q}_\mathbf{m}^\mathbf{0} X_0^\alpha$ and from $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$ to $\mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$, respectively.*

The proof of this theorem is based in the following extension of the Contraction Mapping Theorem, see [22].

Lemma 2.2.5. *Let X and Y be complete metric spaces with metrics d_x and d_y. Let $H : X \times Y \to X \times Y$ be a continuous function satisfying the following:*

(1) $H(x, y) = (F(x), G(x, y))$, F does not depend on y.

(2) There is a constant θ with $0 \leq \theta < 1$ such that one has

$$d_x(F(x_1), F(x_2)) \leq \theta d_x(x_1, x_2), \qquad x_1, x_2 \in X,$$
$$d_y(G(x, y_1), G(x, y_2)) \leq \theta d_y(y_1, y_2), \qquad x \in X, y_1, y_2 \in Y.$$

Then there is a unique fixed point (x^*, y^*) of H. Moreover, if (x_n, y_n) is any sequence of iterations,

$$(x_{n+1}, y_{n+1}) = H(x_n, y_n) \qquad \text{for} \quad n \geq 1,$$

then

$$\lim_{n \to \infty} (x_n, y_n) = (x^*, y^*).$$

In [22] and [52] the authors use this lemma to show the existence of an appropriate fixed point which will give the desired differentiability. In our case, we consider the maps $\mathbf{\Pi_0^\varepsilon} : \tilde{\mathcal{F}}_0(L,R) \times \mathcal{E}_0 \to \tilde{\mathcal{F}}_0(L,R) \times \mathcal{E}_0$ and $\mathbf{\Pi_\varepsilon} : \tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon \to \tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon$ given by

$$\mathbf{\Pi_0^\varepsilon} : (\chi_0^\varepsilon, \Upsilon_0^\varepsilon) \to (\mathbf{T_0^\varepsilon}\chi_0^\varepsilon, \mathbf{D}_0^\varepsilon(\chi_0^\varepsilon, \Upsilon_0^\varepsilon)),$$

and

$$\mathbf{\Pi_\varepsilon} : (\chi_\varepsilon, \Upsilon_\varepsilon) \to (\mathbf{T_\varepsilon}\chi_\varepsilon, \mathbf{D}_\varepsilon(\chi_\varepsilon, \Upsilon_\varepsilon)),$$

where

$$\tilde{\mathcal{F}}_\varepsilon(L,R) = \Big\{ \chi_\varepsilon : \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha \to \mathbf{Q_m^\varepsilon} X_\varepsilon^\alpha \ / \ \|\chi_\varepsilon(p) - \chi_\varepsilon(p')\|_{X_\varepsilon^\alpha} \le L\|p-p'\|_{X_\varepsilon^\alpha}, \ \ p, p' \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha,$$

$$\operatorname{supp}(\chi_\varepsilon) \subset \{\phi \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha, \|\phi\|_{X_\varepsilon^\alpha} \le R\}\Big\}, \qquad 0 \le \varepsilon \le \varepsilon_0$$

and

$$\mathcal{E}_\varepsilon = \{\Upsilon_\varepsilon : \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha \to \mathcal{L}(\mathbf{P_m^\varepsilon} X_\varepsilon^\alpha, \mathbf{Q_m^\varepsilon} X_\varepsilon^\alpha) \text{ continuous :}$$

$$\|\Upsilon_\varepsilon(p)p'\|_{X_\varepsilon^\alpha} \le \|p'\|_{X_\varepsilon^\alpha}, \quad p, p' \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha\} \qquad 0 \le \varepsilon \le \varepsilon_0.$$

Notice that the last contiditon in the definition of $\mathcal{E}_\varepsilon$ could be written equivalently as $\|\Upsilon(p)\|_{\mathcal{L}} \le 1$ for all $p \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha$.

Recall that $\mathbf{T_0^\varepsilon}$, $\mathbf{T_\varepsilon}$ are the functionals described in (2.1.30) and (2.1.31), involved in the proof of the existence of the inertial manifold mentioned in Section 2.1.3, that is

$$(\mathbf{T_0^\varepsilon}\chi_0^\varepsilon)(\xi) = \int_{-\infty}^0 e^{A_\varepsilon \mathbf{Q_m^0} s} \mathbf{Q_m^0} F_0^\varepsilon(u_0^\varepsilon(s))ds, \tag{2.2.4}$$

$$(\mathbf{T_\varepsilon}\chi_\varepsilon)(\eta) = \int_{-\infty}^0 e^{A_\varepsilon \mathbf{Q_m^\varepsilon} s} \mathbf{Q_m^\varepsilon} F_\varepsilon(u_\varepsilon(s))ds, \tag{2.2.5}$$

with $u_0^\varepsilon(t) = p_0^\varepsilon(t) + \chi_0^\varepsilon(p_0^\varepsilon(t))$, $u_\varepsilon(t) = p_\varepsilon(t) + \chi_\varepsilon(p_\varepsilon(t))$ and $p_0^\varepsilon(\cdot)$, $p_\varepsilon(\cdot)$ the solutions of (2.1.32), (2.1.33).

The functionals, $\mathbf{D_0^\varepsilon}(\chi_0^\varepsilon, \Upsilon_0^\varepsilon)$, $\mathbf{D_\varepsilon}(\chi_\varepsilon, \Upsilon_\varepsilon)$ are given as follows: for any $\xi \in \mathbf{P_m^0} X_0^\alpha$, $\eta \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha$,

$$\mathbf{D_0^\varepsilon}(\chi_0^\varepsilon, \Upsilon_0^\varepsilon)(\xi) = \int_{-\infty}^0 e^{A_0 \mathbf{Q_m^0} s} \mathbf{Q_m^0} DF_0^\varepsilon(u_0^\varepsilon(s))(I + \Upsilon_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(\xi, s)ds, \tag{2.2.6}$$

and

$$\mathbf{D_\varepsilon}(\chi_\varepsilon, \Upsilon_\varepsilon)(\eta) = \int_{-\infty}^0 e^{A_\varepsilon \mathbf{Q_m^\varepsilon} s} \mathbf{Q_m^\varepsilon} DF_\varepsilon(u_\varepsilon(s))(I + \Upsilon_\varepsilon(p_\varepsilon(s)))\Theta_\varepsilon(\eta, s)ds, \tag{2.2.7}$$

with u_0^ε, p_0^ε, u_ε, p_ε as above and moreover, $\Theta_0^\varepsilon(\xi,t) = \Theta_0^\varepsilon(\chi_0^\varepsilon, \Upsilon_0^\varepsilon, \xi, t)$, $\Theta_\varepsilon(\eta,t) = \Theta_\varepsilon(\chi_\varepsilon, \Upsilon_\varepsilon, \eta, t)$ are the linear maps from $\mathbf{P_m^0} X_0^\alpha$ to $\mathbf{P_m^0} X_0^\alpha$ and from $\mathbf{P_m^\varepsilon} X_\varepsilon^\alpha$ to $\mathbf{P_m^\varepsilon} X_\varepsilon^\alpha$ satisfying

$$\begin{cases} \Theta_t = -A_0\Theta + \mathbf{P_m^0} DF_0^\varepsilon(u_0^\varepsilon(t))(I + \Upsilon_0^\varepsilon(p_0^\varepsilon(t)))\Theta \\ \Theta(\xi, 0) = I, \end{cases} \tag{2.2.8}$$

and

$$\begin{cases} \Theta_t = -A_\varepsilon\Theta + \mathbf{P_m^\varepsilon} DF_\varepsilon(u_\varepsilon(t))(I + \Upsilon_\varepsilon(p_\varepsilon(t)))\Theta \\ \Theta(\eta, 0) = I, \end{cases} \tag{2.2.9}$$

respectively.

In fact, in these works it is obtained that the fixed point $(\chi_0^{\varepsilon *}, \Upsilon_0^{\varepsilon *}) = (\Psi_0^\varepsilon, D\Psi_0^\varepsilon)$, $(\chi_\varepsilon^*, \Upsilon_\varepsilon^*) = (\Psi_\varepsilon, D\Psi_\varepsilon)$ with Ψ_0^ε and Ψ_ε the inertial manifolds given by the fixed points of the functionals $\mathbf{T_0^\varepsilon}$ and $\mathbf{T_\varepsilon}$ and $D\Psi_0^\varepsilon$, $D\Psi_\varepsilon$ are the Frechet derivatives of the inertial manifolds.

In order to prove the $C^{1,\theta}$ smoothness of the inertial manifolds Φ_0^ε, Φ_ε, we will show that if we denote the set

$$\mathcal{E}_\varepsilon^{\theta,M} = \{\Upsilon_\varepsilon \in \mathcal{E}_\varepsilon : \|\Upsilon_\varepsilon(p) - \Upsilon_\varepsilon(p')\|_{\mathcal{L}} \leq M\|p - p'\|_{X_\varepsilon^\alpha}^\theta, \quad \forall p, p' \in \mathbf{P_m^\varepsilon} X_\varepsilon^\alpha\}$$

which is a closed set in $\mathcal{E}_\varepsilon$, then there exist appropriate θ and M such that the maps $\mathbf{D_0^\varepsilon}(\Psi_0^\varepsilon, \cdot)$ and $\mathbf{D_\varepsilon}(\Psi_\varepsilon, \cdot)$ from (2.2.6) and (2.2.7) with Ψ_0^ε, Ψ_ε the obtained inertial manifolds, transform $\mathcal{E}_\varepsilon^{\theta,M}$ into itself, see Lemma 2.2.9 below, which will imply that the fixed point of the maps $\mathbf{\Pi_0^\varepsilon}$ and $\mathbf{\Pi_\varepsilon}$ lie in $\tilde{\mathcal{F}}_0(L,R) \times \mathcal{E}_0^{\theta,M}$ and $\tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon^{\theta,M}$, respectively, obtaining the desired regularity.

Throughout this subsection, we provide a proof of Proposition 2.2.1 for the inertial manifold Φ_ε for each $\varepsilon > 0$. Note that the proof of this result for the inertial manifold Φ_0^ε, consists in following, step by step, the same way. Then, we focus now in the inertial manifold Φ_ε with $\varepsilon > 0$ fixed.

We start with some estimates.

Lemma 2.2.6. *Let $p_\varepsilon^1(t)$ and $p_\varepsilon^2(t)$ be solutions of (2.1.33) with $p_\varepsilon^1(0)$ and $p_\varepsilon^2(0)$ its initial data, respectively. Then, for $t \leq 0$,*

$$\|p_\varepsilon^1(t) - p_\varepsilon^2(t)\|_{X_\varepsilon^\alpha} \leq \|p_\varepsilon^1(0) - p_\varepsilon^2(0)\|_{X_\varepsilon^\alpha} e^{-[2L_F(\lambda_m^\varepsilon)^\alpha + \lambda_m^\varepsilon]t}$$

Proof. By the variation of constants formula,

$$p_\varepsilon^1(t) - p_\varepsilon^2(t) = e^{-A_\varepsilon t}[p_\varepsilon^1(0) - p_\varepsilon^2(0)] +$$

$$+ \int_0^t e^{-A_\varepsilon(t-s)} \mathbf{P_m^\varepsilon}[F_\varepsilon(p_\varepsilon^1(s) + \Psi_\varepsilon(p_\varepsilon^1(s))) - F_\varepsilon(p_\varepsilon^2(s) + \Psi_\varepsilon(p_\varepsilon^2(s)))]ds.$$

Hence, applying (2.1.40) and (2.1.41) and taking into account that $\Psi_\varepsilon, F_\varepsilon$ are uniformly Lipschitz with Lipschitz constants $L < 1$ and L_F, respectively, we get

$$\|p_\varepsilon^1(t)-p_\varepsilon^2(t)\|_{X_\varepsilon^\alpha} \leq e^{-\lambda_m^\varepsilon t}\|p_\varepsilon^1(0)-p_\varepsilon^2(0)\|_{X_\varepsilon^\alpha}+2L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)}\|p_\varepsilon^1(s)-p_\varepsilon^2(s)\|_{X_\varepsilon^\alpha} ds.$$

By Gronwall inequality,

$$\|p_\varepsilon^1(t) - p_\varepsilon^2(t)\|_{X_\varepsilon^\alpha} \leq \|p_\varepsilon^1(0) - p_\varepsilon^2(0)\|_{X_\varepsilon^\alpha} e^{-[2L_F(\lambda_m^\varepsilon)^\alpha+\lambda_m^\varepsilon]t},$$

as we wanted to prove. ∎

Lemma 2.2.7. *Let $\Psi_\varepsilon \in \tilde{\mathcal{F}}_\varepsilon(L,R)$ with $L<1$ and $\Upsilon_\varepsilon \in \mathcal{E}_\varepsilon$, $0<\varepsilon\leq\varepsilon_0$. Then, for $t\leq 0$,*

$$\|\Theta_\varepsilon(p_\varepsilon^0,t)\|_{\mathcal{L}} \leq e^{-[2L_F(\lambda_m^\varepsilon)^\alpha+\lambda_m^\varepsilon]t}.$$

Proof. If $z_\varepsilon \in \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$, with the aid of the variation of constants formula applied to (2.2.9), we have for $t\leq 0$,

$$\|\Theta_\varepsilon(p_\varepsilon^0,t)z_\varepsilon\|_{X_\varepsilon^\alpha} \leq \|e^{-A_\varepsilon \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} t} z_\varepsilon\|_{X_\varepsilon^\alpha} +$$

$$+ \int_t^0 \left\| e^{-A_\varepsilon \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(t-s)} \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} DF_\varepsilon(u_\varepsilon(s))(I+\Upsilon_\varepsilon(p_\varepsilon(s)))\Theta_\varepsilon(p_\varepsilon^0,s)z_\varepsilon \right\|_{X_\varepsilon^\alpha} ds.$$

Hence as before,

$$\|\Theta_\varepsilon(p_\varepsilon^0,t)z_\varepsilon\|_{X_\varepsilon^\alpha} \leq e^{-\lambda_m^\varepsilon t}\|z_\varepsilon\|_{X_\varepsilon^\alpha} + 2L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)}\|\Theta_\varepsilon(p_\varepsilon^0,s)z_\varepsilon\|_{X_\varepsilon^\alpha}.$$

Using Gronwall inequality, we get

$$\|\Theta_\varepsilon(p_\varepsilon^0,t)z_\varepsilon\|_{X_\varepsilon^\alpha} \leq e^{-[2L_F(\lambda_m^\varepsilon)^\alpha+\lambda_m^\varepsilon]t}\|z_\varepsilon\|_{X_\varepsilon^\alpha}$$

from where we get the result. ∎

Lemma 2.2.8. *Let $0<\theta\leq\theta_F$ and $M>0$ fixed. Let $p_\varepsilon^1, p_\varepsilon^2 \in \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha$ and consider $\Theta_\varepsilon^1(t) = \Theta_\varepsilon(p_\varepsilon^1,t)$, $\Theta_\varepsilon^2(t) = \Theta_\varepsilon(p_\varepsilon^2,t)$ the solutions of (2.2.9) for some $\Upsilon_\varepsilon \in \mathcal{E}_\varepsilon^{\theta,M}$. Then, for $t\leq 0$,*

$$\|\Theta_\varepsilon^1(t)-\Theta_\varepsilon^2(t)\|_{\mathcal{L}} \leq \left(\frac{2L}{(\theta+1)L_F}+\frac{M}{2(\theta+1)}\right)\|p_\varepsilon^1-p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta \; e^{-(2(\theta+2)L_F(\lambda_m^\varepsilon)^\alpha+(\theta+1)\lambda_m^\varepsilon)t}.$$

Proof. Applying the variation of constants formula to (2.2.9), for $t\leq 0$,

$$\|\Theta_\varepsilon^1(t)-\Theta_\varepsilon^2(t)\|_{\mathcal{L}} \leq \int_t^0 \Big\| e^{-A_\varepsilon \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}(t-s)} \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} [DF_\varepsilon(u_\varepsilon^1(s))(I+\Upsilon_\varepsilon(p_\varepsilon^1(s)))\Theta_\varepsilon^1(s)$$

$$-DF_\varepsilon(u_\varepsilon^2(s))(I+\Upsilon_\varepsilon(p_\varepsilon^2(s)))\Theta_\varepsilon^2(s)]\Big\|_{\mathcal{L}} ds$$

with $u_\varepsilon^i(s) = p_\varepsilon^i(s)+\Psi_\varepsilon(p_\varepsilon^i(s))$, $i=1,2$.

We can decompose it in the following way,

$$\|\Theta_\varepsilon^1(t)-\Theta_\varepsilon^2(t)\|_{\mathcal{L}} \leq$$

$$\leq \int_t^0 \left\| e^{-A_\varepsilon \mathbf{P}_{\mathbf{m}}^{\varepsilon}(t-s)} \mathbf{P}_{\mathbf{m}}^{\varepsilon} [DF_\varepsilon(u_\varepsilon^1(s)) - DF_\varepsilon(u_\varepsilon^2(s))](I+\Upsilon_\varepsilon(p_\varepsilon^1(s)))\Theta_\varepsilon^1(s) \right\|_{\mathcal{L}} ds +$$

$$+ \int_t^0 \left\| e^{-A_\varepsilon \mathbf{P}_{\mathbf{m}}^{\varepsilon}(t-s)} \mathbf{P}_{\mathbf{m}}^{\varepsilon} DF_\varepsilon(u_\varepsilon^2(s))(\Upsilon_\varepsilon(p_\varepsilon^1(s)) - \Upsilon_\varepsilon(p_\varepsilon^2(s)))\Theta_\varepsilon^1(s) \right\|_{\mathcal{L}} ds +$$

$$+ \int_t^0 \left\| e^{-A_\varepsilon \mathbf{P}_{\mathbf{m}}^{\varepsilon}(t-s)} \mathbf{P}_{\mathbf{m}}^{\varepsilon} DF_\varepsilon(u_\varepsilon^2(s))[(I+\Upsilon_\varepsilon(p_\varepsilon^2(s)))(\Theta_\varepsilon^1(s)-\Theta_\varepsilon^2(s)) \right\|_{\mathcal{L}} ds =$$

$$= I_1 + I_2 + I_3.$$

We analyze each term separately.

By hipothesis **(H2')**, (2.1.41) and Lemma 2.2.7,

$$I_1 \leq 2L(\lambda_m^\varepsilon)^\alpha e^{-\lambda_m^\varepsilon t} \int_t^0 \|u_\varepsilon^1(s)-u_\varepsilon^2(s)\|_{X_\varepsilon^\alpha}^\theta e^{-2L_F(\lambda_m^\varepsilon)^\alpha s} ds \leq$$

$$\leq 4L(\lambda_m^\varepsilon)^\alpha e^{-\lambda_m^\varepsilon t} \int_t^0 \|p_\varepsilon^1(s)-p_\varepsilon^2(s)\|_{X_\varepsilon^\alpha}^\theta e^{-2L_F(\lambda_m^\varepsilon)^\alpha s} ds.$$

Applying Lemma 2.2.6,

$$I_1 \leq \frac{2L}{(\theta+1)L_F} \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta e^{-[2(\theta+1)L_F(\lambda_m^\varepsilon)^\alpha + (\theta+1)\lambda_m^\varepsilon]t}.$$

Since $\Upsilon_\varepsilon \in \mathcal{E}_\varepsilon^{\theta,M}$, $0<\theta\leq\theta_F$, and by Lemma 2.2.7, we have

$$I_2 \leq L_F(\lambda_m^\varepsilon)^\alpha M e^{-\lambda_m^\varepsilon t} \int_t^0 \|p_\varepsilon^1(s)-p_\varepsilon^2(s)\|_{X_\varepsilon^\alpha}^\theta e^{-2L_F(\lambda_m^\varepsilon)^\alpha s} ds.$$

Applying Lemma 2.2.6,

$$I_2 \leq \frac{M}{2(\theta+1)} \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta e^{-[2(\theta+1)L_F(\lambda_m^\varepsilon)^\alpha + (\theta+1)\lambda_m^\varepsilon]t},$$

and the last one,

$$I_3 \leq 2L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)} \|\Theta_\varepsilon^1(s)-\Theta_\varepsilon^2(s)\|_{\mathcal{L}} ds.$$

So,

$$\|\Theta_\varepsilon^1(t)-\Theta_\varepsilon^2(t)\|_{\mathcal{L}} \leq$$

$$\left(\frac{2L}{(\theta+1)L_F} + \frac{M}{2(\theta+1)}\right) \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta e^{-[2(\theta+1)L_F(\lambda_m^\varepsilon)^\alpha + (\theta+1)\lambda_m^\varepsilon]t}$$

$$+2L_F(\lambda_m^\varepsilon)^\alpha \int_t^0 e^{-\lambda_m^\varepsilon(t-s)} \|\Theta_\varepsilon^1(s)-\Theta_\varepsilon^2(s)\|_{\mathcal{L}} ds.$$

Applying Gronwall inequality,

$$\|\Theta_\varepsilon^1(t)-\Theta_\varepsilon^2(t)\|_{\mathcal{L}} \leq$$
$$\leq \left(\frac{2L}{(\theta+1)L_F}+\frac{M}{2(\theta+1)}\right)\|p_\varepsilon^1-p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta e^{-[2(\theta+2)L_F(\lambda_m^\varepsilon)^\alpha+(\theta+1)\lambda_m^\varepsilon]t},$$

which shows the result. ■

For the sake of notation, there are several exponents that repeat themselves very often and they are kind of long. We will abbreviate the exponents as follows:

$$\begin{aligned}\Lambda_0 &= 2L_F(\lambda_m^\varepsilon)^\alpha+\lambda_m^\varepsilon\\ \Lambda_1 &= \lambda_{m+1}^\varepsilon-(\theta+1)\lambda_m^\varepsilon-2(\theta+1)L_F(\lambda_m^\varepsilon)^\alpha\\ \Lambda_2 &= \lambda_{m+1}^\varepsilon-(\theta+1)\lambda_m^\varepsilon-2(\theta+2)L_F(\lambda_m^\varepsilon)^\alpha\end{aligned} \tag{2.2.10}$$

With these estimates we can prove the following Lemma.

Lemma 2.2.9. *If we choose θ such that $0<\theta\leq\theta_F$ and $\theta<\theta_0$ with θ_0 given by* (2.2.1)*, then there exist $M_0=M_0(\theta)>0$ such that for each $M\geq M_0$ and for ε small enough, we have $\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\cdot)$ maps $\mathcal{E}_\varepsilon^{\theta,M}$ into $\mathcal{E}_\varepsilon^{\theta,M}$.*

Proof. Let $\Upsilon_\varepsilon\in\mathcal{E}_\varepsilon^{\theta,M}$ and $p_\varepsilon^1,p_\varepsilon^2\in\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}X_\varepsilon^\alpha$. In [52] the authors prove $\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\cdot)$ maps $\mathcal{E}_\varepsilon$ into $\mathcal{E}_\varepsilon$. So, it remains to prove that,

$$\|\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\Upsilon_\varepsilon)(p_\varepsilon^1)-\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\Upsilon_\varepsilon)(p_\varepsilon^2)\|_{\mathcal{L}}\leq M\|p_\varepsilon^1-p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta,$$

with M and θ as in the statement.

From expression (2.2.7), we have,

$$\|\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\Upsilon_\varepsilon)(p_\varepsilon^1)-\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\Upsilon_\varepsilon)(p_\varepsilon^2)\|_{\mathcal{L}}\leq$$
$$\int_{-\infty}^0\left\|e^{A_\varepsilon\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}s}\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}[DF_\varepsilon(u_\varepsilon^1(s))(I+\Upsilon_\varepsilon(p_\varepsilon^1(s)))\Theta_\varepsilon^1(s)-DF_\varepsilon(u_\varepsilon^2(s))(I+\Upsilon_\varepsilon(p_\varepsilon^2(s)))\Theta_\varepsilon^2(s)]\right\|_{\mathcal{L}}ds,$$

with $p_\varepsilon^i(s)$ the solution of (2.1.33) with $p_\varepsilon^i(0)=p_\varepsilon^i$ and $u_\varepsilon^i(s)=p_\varepsilon^i(s)+\Psi_\varepsilon(p_\varepsilon^i(s))$, for $i=1,2$.

In a similar way as in proof of Lemma 2.2.8, we decompose it as follows,

$$\|\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\Upsilon_\varepsilon)(p_\varepsilon^1)-\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon,\Upsilon_\varepsilon)(p_\varepsilon^2)\|_{\mathcal{L}}\leq$$
$$\begin{aligned}&\leq\int_{-\infty}^0\left\|e^{A_\varepsilon\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}s}\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}[DF_\varepsilon(u_\varepsilon^1(s))-DF_\varepsilon(u_\varepsilon^2(s))](I+\Upsilon_\varepsilon(p_\varepsilon^1(s)))\Theta_\varepsilon^1(s)\right\|_{\mathcal{L}}ds+\\ &\quad+\int_{-\infty}^0\left\|e^{A_\varepsilon\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}s}\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}DF_\varepsilon(u_\varepsilon^2(s))[\Upsilon_\varepsilon(p_\varepsilon^1(s))-\Upsilon_\varepsilon(p_\varepsilon^2(s))]\Theta_\varepsilon^1(s)\right\|_{\mathcal{L}}ds+\\ &\quad+\int_{-\infty}^0\left\|e^{A_\varepsilon\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}s}\mathbf{Q}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}DF_\varepsilon(u_\varepsilon^2(s))(I+\Upsilon_\varepsilon(p_\varepsilon^2(s)))[\Theta_\varepsilon^1(s)-\Theta_\varepsilon^2(s)]\right\|_{\mathcal{L}}ds=\end{aligned}$$

$$= I_1 + I_2 + I_3.$$

Following the same arguments used in that proof and since $\Upsilon_\varepsilon \in \mathcal{E}_\varepsilon^{\theta,M}$ we get

$$I_1 \le 4L(\lambda_{m+1}^\varepsilon)^\alpha \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta \int_{-\infty}^0 e^{\Lambda_1 s} ds \le \frac{4L(\lambda_{m+1}^\varepsilon)^\alpha}{\Lambda_1} \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta$$

Similarly, for I_2,

$$I_2 \le L_F(\lambda_{m+1}^\varepsilon)^\alpha M \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta \int_{-\infty}^0 e^{\Lambda_1 s} ds \le \frac{L_F(\lambda_{m+1}^\varepsilon)^\alpha M}{\Lambda_1} \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta.$$

And finally, applying Lemma 2.2.8,

$$I_3 \le 2L_F(\lambda_{m+1}^\varepsilon)^\alpha \left(\frac{2L}{(\theta+1)L_F} + \frac{M}{2(\theta+1)}\right) \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta \int_{-\infty}^0 e^{-\Lambda_2 s} ds$$

which implies,

$$I_3 \le \frac{2L_F(\lambda_{m+1}^\varepsilon)^\alpha}{\Lambda_2} \left(\frac{2L}{(\theta+1)L_F} + \frac{M}{2(\theta+1)}\right) \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta.$$

Putting everything together we obtain

$$\|\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon, \Upsilon_\varepsilon)(p_\varepsilon^1) - \mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon, \Upsilon_\varepsilon)(p_\varepsilon^2)\|_{\mathcal{L}} \le$$
$$(4L + ML_F)(\lambda_{m+1}^\varepsilon)^\alpha \Big(\frac{1}{\Lambda_1} + \frac{1}{(\theta+1)\Lambda_2}\Big) \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta$$

But since $\Lambda_2 \le \Lambda_1$, see (2.2.10), and $\theta > 0$, we have

$$\|\mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon, \Upsilon_\varepsilon)(p_\varepsilon^1) - \mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon, \Upsilon_\varepsilon)(p_\varepsilon^2)\|_{\mathcal{L}} \le (4L + ML_F)(\lambda_{m+1}^\varepsilon)^\alpha \frac{2}{\Lambda_2} \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta$$
$$= \Big(\frac{8L(\lambda_{m+1}^\varepsilon)^\alpha}{\Lambda_2} + M\frac{2L_F(\lambda_{m+1}^\varepsilon)^\alpha}{\Lambda_2}\Big) \|p_\varepsilon^1 - p_\varepsilon^2\|_{X_\varepsilon^\alpha}^\theta$$

But if we consider

$$\theta^0 = \frac{\lambda_{m+1}^0 - \lambda_m^0 - 4L_F(\lambda_m^0)^\alpha - 2L_F(\lambda_{m+1}^0)^\alpha}{2L_F(\lambda_m^0)^\alpha + \lambda_m^0},$$

then, direct computations show that if $\theta < \theta_0$ and ε is small, then $\frac{2L_F(\lambda_{m+1}^\varepsilon)^\alpha}{\Lambda_2} \le \eta$ for some $\eta < 1$. This implies that if we choose M large enough then

$$\Big(\frac{8L(\lambda_{m+1}^\varepsilon)^\alpha}{\Lambda_2} + M\frac{2L_F(\lambda_{m+1}^\varepsilon)^\alpha}{\Lambda_2}\Big) \le M$$

which shows the result. ∎

We can prove now the main result of this subsection.

Proof of Proposition 2.2.1. Since $\Phi_\varepsilon := \Psi_\varepsilon \circ j_\varepsilon^{-1}$ and j_ε is an isomorphism, see (2.1.8), to prove $\Phi_\varepsilon \in C^{1,\theta}(\mathbb{R}^m, X_\varepsilon^\alpha)$ for some θ, is equivalent to prove $\Psi_\varepsilon \in C^{1,\theta}(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha, X_\varepsilon^\alpha)$.

In [52], the authors prove the existence of the unique fixed point $(\Psi_\varepsilon^*, \Upsilon_\varepsilon^*) = (\Psi_\varepsilon, D\Psi_\varepsilon) \in \tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon$ of the map

$$\mathbf{\Pi}_{\boldsymbol{\varepsilon}} : (\Psi_\varepsilon, \Upsilon_\varepsilon) \to (\mathbf{T}_{\boldsymbol{\varepsilon}}\Psi_\varepsilon, \mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon, \Upsilon_\varepsilon)).$$

We want to prove this fixed point, in fact, belongs to $\tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon^{\theta,M}$. We proceed as follows. Let $\{z_n\}_{n\geq 0} \in \tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon^{\theta,M}$ a sequence given by

$$z_0 = (\Psi_\varepsilon, 0), \qquad z_1 = (\mathbf{T}_{\boldsymbol{\varepsilon}}\Psi_\varepsilon, \mathbf{D}_{\boldsymbol{\varepsilon}}(\Psi_\varepsilon, 0)), \quad \ldots \quad z_n = (\mathbf{T}_{\boldsymbol{\varepsilon}}^{(n)}(\Psi_\varepsilon), \mathbf{D}_{\boldsymbol{\varepsilon}}{}^{(n)}(\Psi_\varepsilon, 0)).$$

Note that by Lemma 2.2.9, $\{z_n\}_{n\geq 0} \in \tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon^{\theta,M}$ with θ and M described in this lemma.

Since Ψ_ε is the fixed point of $\mathbf{T}_{\boldsymbol{\varepsilon}}$, then,

$$z_n = (\Psi_\varepsilon, \mathbf{D}_{\boldsymbol{\varepsilon}}{}^{(n)}(\Psi_\varepsilon, 0)), \qquad \forall n \geq 0.$$

By Lemma 2.2.5,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (\Psi_\varepsilon, \mathbf{D}_{\boldsymbol{\varepsilon}}{}^{(n)}(\Psi_\varepsilon, 0)) = (\Psi_\varepsilon, D\Psi_\varepsilon).$$

Hence, since $\mathcal{E}_\varepsilon^{\theta,M}$ is a closed subspace of $\mathcal{E}_\varepsilon$, then

$$\lim_{n\to\infty} z_n = (\Psi_\varepsilon, D\Psi_\varepsilon) \in \tilde{\mathcal{F}}_\varepsilon(L,R) \times \mathcal{E}_\varepsilon^{\theta,M}.$$

That is, $\Psi_\varepsilon \in C^{1,\theta}(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} X_\varepsilon^\alpha, X_\varepsilon^\alpha)$, for $0 < \varepsilon \leq \varepsilon_0$, with $0 < \theta \leq \theta_F$ and $\theta < \theta^0$, see (2.2.1). Then, $\Phi_\varepsilon \in C^{1,\theta}(\mathbb{R}^m, X_\varepsilon^\alpha)$ as we wanted to prove. ■

2.2.3. $C^{1,\theta}$-convergence of inertial manifolds

In this subsection we study the $C^{1,\theta}$-convergence, with $0 < \theta \leq 1$ small enough, of the inertial manifolds Φ_0^ε, Φ_ε, $0 < \varepsilon \leq \varepsilon_0$. For that we will obtain first the C^1-convergence of these manifolds, and, with an interpolation argument and applying the results obtained in the previous subsection, we get the $C^{1,\theta}$-convergence and a rate of this convergence.

Before proving the main result of this subsection, Theorem 2.2.2, we need the following estimate.

Lemma 2.2.10. *Let* $\Theta_0^\varepsilon(j_0^{-1}(z),t) = \Theta_0^\varepsilon(\Psi_0^\varepsilon, D\Psi_0^\varepsilon, j_0^{-1}(z),t)$ *and* $\Theta_\varepsilon(j_\varepsilon^{-1}(z),t) = \Theta_\varepsilon(\Psi_\varepsilon, D\Psi_\varepsilon, j_\varepsilon^{-1}(z),t)$ *be solutions of (2.2.8) and (2.2.9), for* $z\in\mathbb{R}^m$ *and* $t\leq 0$. *Then, we have,*

$$\|\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\Theta_0^\varepsilon(j_0^{-1}(z),t) - \Theta_\varepsilon(j_\varepsilon^{-1}(z),t)\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\|_{\mathcal{L}(\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}X_0^\alpha, \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}X_\varepsilon^\alpha)} \leq$$

$$C[\beta(\varepsilon) + [\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)]^\theta] e^{-[(4+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + (\theta+1)\lambda_m^\varepsilon + 3\theta]t} +$$

$$+\frac{\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\|_\infty}{2} e^{-[4L_F(\lambda_m^\varepsilon)^\alpha + \lambda_m^\varepsilon]t},$$

where C *is a constant independent of* ε, $0<\theta\leq\theta_F$ *and* $\theta<\theta_0$, *and* κ *is given by* (2.1.3).

Remark 2.2.11. *We denote by* $\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\|_\infty$ *the sup norm,*

$$\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\|_\infty = \sup_{p\in \mathbf{P}_{\mathbf{m}}^{\mathbf{0}}X_0^\alpha} \|ED\Psi_0^\varepsilon(p) - D\Psi_\varepsilon(\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} Ep)\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\|_{\mathcal{L}(\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}X_0^\alpha, X_\varepsilon^\alpha)}, \tag{2.2.11}$$

Proof. With the Variation of Constants Formula applied to (2.2.8) and (2.2.9), and denoting by $\Theta_0^\varepsilon(t) = \Theta_0^\varepsilon(j_0^{-1}(z),t)$ and $\Theta_\varepsilon(t) = \Theta_\varepsilon(j_\varepsilon^{-1}(z),t)$, we get

$$E\Theta_0^\varepsilon(t) - \Theta_\varepsilon(t)\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E = Ee^{-A_0\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}t} - e^{-A_\varepsilon \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}t}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E +$$

$$+\int_t^0 \Big(Ee^{-A_0\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}(t-s)}\mathbf{P}_{\mathbf{m}}^{\mathbf{0}} DF_0^\varepsilon(u_0^\varepsilon(s))(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s) -$$

$$e^{-A_\varepsilon\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(t-s)}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} DF_\varepsilon(u_\varepsilon(s))(I + D\Psi_\varepsilon(p_\varepsilon(s)))\Theta_\varepsilon(s)\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\Big)\, ds =$$

$$= I' + \int_t^0 I$$

We estimate now I' and I. Notice first that $\|I'\|_{\mathcal{L}(\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}X_0^\alpha, \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}X_\varepsilon^\alpha)}$ is analyzed with Lemma 2.1.18.

Moreover, for I we get, the following decomposition:

$$I = Ee^{-A_0\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}(t-s)}\mathbf{P}_{\mathbf{m}}^{\mathbf{0}} DF_0^\varepsilon(u_0^\varepsilon(s))(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s) -$$

$$e^{-A_\varepsilon\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(t-s)}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} DF_\varepsilon(u_\varepsilon(s))(I + D\Psi_\varepsilon(p_\varepsilon(s)))\Theta_\varepsilon(s)\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E =$$

$$= \Big(Ee^{-A_0\mathbf{P}_{\mathbf{m}}^{\mathbf{0}}(t-s)}\mathbf{P}_{\mathbf{m}}^{\mathbf{0}} - e^{-A_\varepsilon\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(t-s)}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} E\Big) DF_0^\varepsilon(u_0^\varepsilon(s))(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s)$$

$$+e^{-A_\varepsilon\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}(t-s)}\mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}\Big(EDF_0^\varepsilon(u_0^\varepsilon(s)) - DF_\varepsilon(Eu_0^\varepsilon(s))E\Big)(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s)$$

$$+e^{-A_\varepsilon \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(t-s)}\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}\Big(DF_\varepsilon(Eu^\varepsilon_0(s)) - DF_\varepsilon(u_\varepsilon(s))\Big)E(I + D\Psi^\varepsilon_0(p^\varepsilon_0(s)))\Theta^\varepsilon_0(s)$$

$$+e^{-A_\varepsilon \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(t-s)}\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}DF_\varepsilon(u_\varepsilon(s))\Big(E(I + D\Psi^\varepsilon_0(p^\varepsilon_0(s))) - (I + D\Psi_\varepsilon(\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}Ep^\varepsilon_0(s)))E\Big)\Theta^\varepsilon_0(s)$$

$$+e^{-A_\varepsilon \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(t-s)}\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}DF_\varepsilon(u_\varepsilon(s))\Big((I + D\Psi_\varepsilon(\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}Ep^\varepsilon_0(s))) - (I + D\Psi_\varepsilon(p_\varepsilon(s)))\Big)E\Theta^\varepsilon_0(s)$$

$$+e^{-A_\varepsilon \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(t-s)}\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}DF_\varepsilon(u_\varepsilon(s))(I + D\Psi_\varepsilon(p_\varepsilon(s)))\Big(E\Theta^\varepsilon_0(s) - \Theta_\varepsilon(s)\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E\Big)$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

Now we can study the norm $\|I\|_{\mathcal{L}(\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}X^\alpha_0, \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}X^\alpha_\varepsilon)}$ analyzing the norm of each term separately.

By Lemmas 2.1.18 and 2.2.7 we have,

$$\|I_1\|_{\mathcal{L}(\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}X^\alpha_0, \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}X^\alpha_\varepsilon)} \leq 2L_F C_4 \tau(\varepsilon) e^{-(\lambda^0_m+1)t} e^{(-2L_F(\lambda^0_m)^\alpha+1)s}.$$

With the definition of $\beta(\varepsilon)$ from (2.2.2) and again Lemma 2.2.7,

$$\|I_2\|_{\mathcal{L}(\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}X^\alpha_0, \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}X^\alpha_\varepsilon)} \leq 2(\lambda^\varepsilon_m)^\alpha \beta(\varepsilon) e^{-\lambda^\varepsilon_m t} e^{-2L_F(\lambda^\varepsilon_m)^\alpha s}.$$

To study the term I_3, again, from (2.2.2), Lemma 2.2.7 and the properties on the norm of extension operator, see (2.1.3), for $0 < \theta \leq \theta_F$,

$$\|I_3\|_{\mathcal{L}(\mathbf{P}^{\mathbf{0}}_{\mathbf{m}}X^\alpha_0, \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}X^\alpha_\varepsilon)} \leq 2\kappa(\lambda^\varepsilon_m)^\alpha L \|Eu^\varepsilon_0(s) - u_\varepsilon(s)\|^\theta_{X^\alpha_\varepsilon} e^{-\lambda^\varepsilon_m t} e^{-2L_F(\lambda^\varepsilon_m)^\alpha s}.$$

Remember that,

$$u^\varepsilon_0(s) = p^\varepsilon_0(s) + \Psi^\varepsilon_0(p^\varepsilon_0(s)) = p^\varepsilon_0(s) + \Phi^\varepsilon_0(j_0(p^\varepsilon_0(s))),$$

and for $0 < \varepsilon \leq \varepsilon_0$,

$$u_\varepsilon(s) = p_\varepsilon(s) + \Psi_\varepsilon(p_\varepsilon(s)) = p_\varepsilon(s) + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s))).$$

Then,

$$\|Eu^\varepsilon_0(s) - u_\varepsilon(s)\|^\theta_{X^\alpha_\varepsilon} \leq$$
$$\big(\|p_\varepsilon(s) - Ep^\varepsilon_0(s)\|_{X^\alpha_\varepsilon} + \|\Phi_\varepsilon(j_\varepsilon(p_\varepsilon(s))) - \Phi_\varepsilon(j_0(p^\varepsilon_0(s))\|_{X^\alpha_\varepsilon} +$$
$$\|\Phi_\varepsilon(j_0(p^\varepsilon_0(s)) - \Phi^\varepsilon_0(j_0(p^\varepsilon_0(s)))\|_{X^\alpha_\varepsilon}\big)^\theta \leq$$
$$\big(\|p_\varepsilon(s) - Ep^\varepsilon_0(s)\|_{X^\alpha_\varepsilon} + |j_\varepsilon(p_\varepsilon(s)) - j_0(p^\varepsilon_0(s))|_{0,\alpha} + \|\Phi_\varepsilon - E\Phi^\varepsilon_0\|_{L^\infty(\mathbb{R}^m, X^\alpha_\varepsilon)}\big)^\theta.$$

So, applying Theorem 2.1.4, Lemma 2.1.21, Lemma 2.1.22 and Lemma 2.1.23, and taking into account that $(a+b)^\theta \le a^\theta + b^\theta$ for $0 \le \theta \le 1$, we obtain,

$$\|Eu_0^\varepsilon(s) - u_\varepsilon(s)\|_{X_\varepsilon^\alpha}^\theta \le$$

$$\le (\kappa+2)^\theta \left(\frac{L_F}{(\lambda_m^\varepsilon)^{1-\alpha}} \tau(\varepsilon) |\log(\tau(\varepsilon))| + \rho(\varepsilon) + K_2 e^{-2s}\tau(\varepsilon) \right)^\theta e^{-[(\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha + \lambda_m^\varepsilon]s\theta} +$$
$$+[(\kappa+1)C_P\tau(\varepsilon)(R+C_F)]^\theta e^{-\lambda_m^\varepsilon s\theta} + C[\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)]^\theta \le$$
$$\le C[\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)]^\theta e^{-[(\kappa+2)L_F(\lambda_m^\varepsilon)^\alpha + \lambda_m^\varepsilon + 3]s\theta},$$

with $C > 0$ independent of ε.

Hence,

$$\|I_3\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} \le 2\kappa(\lambda_m^\varepsilon)^\alpha LC[\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)]^\theta e^{-\lambda_m^\varepsilon t} e^{-[(2+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + \theta\lambda_m^\varepsilon + 3\theta]s}.$$

By Lemma 2.2.7, we have,

$$\|I_4\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} \le (\lambda_m^\varepsilon)^\alpha L_F \|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P_m^\varepsilon} E\|_\infty e^{-\lambda_m^\varepsilon t} e^{-2L_F(\lambda_m^\varepsilon)^\alpha s}.$$

By subsection 2.2.2, $D\Psi_\varepsilon \in \mathcal{E}_\varepsilon^{\theta,M}$ for $0 < \theta \le \theta_F$ and $\theta < \theta_0$. Applying estimate (2.1.3), Lemma 2.1.23, and Lemma 2.2.7, we have,

$$\|I_5\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} \le \kappa L_F(\lambda_m^\varepsilon)^\alpha M(\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^\theta e^{-\lambda_m^\varepsilon t} e^{-[(2+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + \theta\lambda_m^\varepsilon + 3\theta]s}$$

Finally, the norm of term I_6 is estimated by,

$$\|I_6\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} \le 2(\lambda_m^\varepsilon)^\alpha L_F e^{-\lambda_m^\varepsilon(t-s)} \|E\Theta_0^\varepsilon(s) - \Theta_\varepsilon(s)\mathbf{P_m^\varepsilon} E\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)}.$$

Putting all together,

$$\|I\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} \le$$
$$CL_F L(\lambda_m^\varepsilon)^\alpha \left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^\theta\right] e^{-\lambda_m^\varepsilon t} e^{-[(2+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + \theta\lambda_m^\varepsilon + 3\theta]s}$$
$$+(\lambda_m^\varepsilon)^\alpha L_F \|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P_m^\varepsilon} E\|_\infty e^{-\lambda_m^\varepsilon t} e^{-2L_F(\lambda_m^\varepsilon)^\alpha s} +$$
$$+2(\lambda_m^\varepsilon)^\alpha L_F e^{-\lambda_m^\varepsilon(t-s)} \|E\Theta_0^\varepsilon(s) - \Theta_\varepsilon(s)\mathbf{P_m^\varepsilon} E\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)}.$$

Then,

$$\|E\Theta_0^\varepsilon(t) - \Theta_\varepsilon(t)\mathbf{P_m^\varepsilon} E\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} \le \|I'\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} + \int_t^0 \|I\|_{\mathcal{L}(\mathbf{P_m^0}X_0^\alpha, \mathbf{P_m^\varepsilon}X_\varepsilon^\alpha)} \le$$

$$\le C_4 e^{-(\lambda_m^0+1)t}\tau(\varepsilon) +$$

$$CL_F L(\lambda_m^\varepsilon)^\alpha \left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^\theta\right] e^{-\lambda_m^\varepsilon t} \int_t^0 e^{-[(2+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + \theta\lambda_m^\varepsilon + 3\theta]s} ds$$

$$+(\lambda_m^\varepsilon)^\alpha L_F \|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P}_\mathbf{m}^\varepsilon E\|_\infty e^{-\lambda_m^\varepsilon t} \int_t^0 e^{-2L_F(\lambda_m^\varepsilon)^\alpha s} ds +$$

$$+2(\lambda_m^\varepsilon)^\alpha L_F e^{-\lambda_m^\varepsilon t} \int_t^0 e^{\lambda_m^\varepsilon s} \|E\Theta_0^\varepsilon(s) - \Theta_\varepsilon(s)\mathbf{P}_\mathbf{m}^\varepsilon E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha, \mathbf{P}_\mathbf{m}^\varepsilon X_\varepsilon^\alpha)} ds.$$

So, we have,

$$\|E\Theta_0^\varepsilon(t) - \Theta_\varepsilon(t)\mathbf{P}_\mathbf{m}^\varepsilon E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha, \mathbf{P}_\mathbf{m}^\varepsilon X_\varepsilon^\alpha)} \leq$$

$$\leq C \left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^\theta\right] e^{-[(2+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + (\theta+1)\lambda_m^\varepsilon + 3\theta]t} +$$

$$+ \frac{\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P}_\mathbf{m}^\varepsilon E\|_\infty}{2} e^{-[2L_F(\lambda_m^\varepsilon)^\alpha + \lambda_m^\varepsilon]t} +$$

$$+2(\lambda_m^\varepsilon)^\alpha L_F e^{-\lambda_m^\varepsilon t} \int_t^0 e^{\lambda_m^\varepsilon s} \|E\Theta_0^\varepsilon(s) - \Theta_\varepsilon(s)\mathbf{P}_\mathbf{m}^\varepsilon E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha, \mathbf{P}_\mathbf{m}^\varepsilon X_\varepsilon^\alpha)} ds.$$

Applying Gronwall inequality,

$$\|E\Theta_0^\varepsilon(t) - \Theta_\varepsilon(t)\mathbf{P}_\mathbf{m}^\varepsilon E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha, \mathbf{P}_\mathbf{m}^\varepsilon X_\varepsilon^\alpha)} \leq$$

$$\leq C \left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^\theta\right] e^{-[(4+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + (\theta+1)\lambda_m^\varepsilon + 3\theta]t} +$$

$$+ \frac{\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon \mathbf{P}_\mathbf{m}^\varepsilon E\|_\infty}{2} e^{-[4L_F(\lambda_m^\varepsilon)^\alpha + \lambda_m^\varepsilon]t},$$

with $C > 0$ a constant independent of ε and $0 < \theta \leq \theta_F$ with $\theta < \theta_0$. ■

We show now the convergence of the differential of inertial manifolds and establish a rate for this convergence. For this, we define θ_1 and $\tilde{\theta}$ as follows,

$$\theta_1 = \frac{\lambda_{m+1}^0 - \lambda_m^0 - 4L_F(\lambda_m^0)^\alpha}{(\kappa+2)L_F(\lambda_m^0)^\alpha + \lambda_m^0 + 3}, \tag{2.2.12}$$

and,

$$\tilde{\theta} = \min\{\theta_F, \theta_0, \theta_1\}. \tag{2.2.13}$$

Proposition 2.2.12. *With Φ_0^ε and Φ_ε the inertial manifolds, and if $\theta < \tilde{\theta}$, we have the following estimate*

$$\|ED\Phi_0^\varepsilon - D\Phi_\varepsilon\|_{C^1(\mathbb{R}^m, X_\varepsilon^\alpha)} \leq C\left[\beta(\varepsilon) + \Big(\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)\Big)^\theta\right] \tag{2.2.14}$$

where C is a constant independent of ε.

Proof. Applying Theorem 2.1.4, it remains to estimate

$$\sup_{z\in\mathbb{R}^m} \|D\Phi_\varepsilon(z) - ED\Phi_0^\varepsilon(z)\|_{\mathcal{L}(\mathbb{R}^m, X_\varepsilon^\alpha)}.$$

That is, we need to estimate $\|D\Phi_\varepsilon - ED\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, X_\varepsilon^\alpha))}$. We know that,

$$\sup_{z\in\mathbb{R}^m} \|ED\Phi_0^\varepsilon(z) - D\Phi_\varepsilon(z)\|_{\mathcal{L}(\mathbb{R}^m, X_\varepsilon^\alpha)} =$$

$$= \sup_{z\in\mathbb{R}^m} \|ED\Psi_0^\varepsilon(j_0^{-1}(z))j_0^{-1} - D\Psi_\varepsilon(j_\varepsilon^{-1}(z))\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E j_0^{-1}\|_{\mathcal{L}(\mathbb{R}^m, X_\varepsilon^\alpha)} =$$

$$= \sup_{z\in\mathbb{R}^m} \|ED\Psi_0^\varepsilon(j_0^{-1}(z)) - D\Psi_\varepsilon(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E j_0^{-1}(z))\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha, X_\varepsilon^\alpha)} =$$

$$= \sup_{p_0^\varepsilon \in \mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha} \|ED\Psi_0^\varepsilon(p_0^\varepsilon) - D\Psi_\varepsilon(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E p_0^\varepsilon)\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0} X_0^\alpha, X_\varepsilon^\alpha)} = \|ED\Psi_0^\varepsilon - D\Psi_\varepsilon E\|_\infty.$$

We have applied $|j_0(p_0^\varepsilon)|_{0,\alpha} = \|p_0^\varepsilon\|_{X_0^\alpha}$ for any $p_0^\varepsilon \in \mathbf{P}_\mathbf{m}^\mathbf{0} X_0$, see (2.1.11).

Then, for $z' \in \mathbb{R}^m$, with the definition (2.2.7), and denoting again by $\Theta_0^\varepsilon(t) = \Theta_0^\varepsilon(j_0^{-1}(z), t)$ and $\Theta_\varepsilon(t) = \Theta_\varepsilon(j_\varepsilon^{-1}(z), t)$, we have

$$ED\Psi_0^\varepsilon(j_0^{-1}(z))j_0^{-1}(z') - D\Psi_\varepsilon(\mathbf{P}_\mathbf{m}^\varepsilon E \circ j_0^{-1}(z))\mathbf{P}_\mathbf{m}^\varepsilon E \circ j_0^{-1}(z') =$$

$$= \int_{-\infty}^0 \Big(E e^{A_0 \mathbf{Q}_\mathbf{m}^\mathbf{0} s} \mathbf{Q}_\mathbf{m}^\mathbf{0} DF_0^\varepsilon(u_0^\varepsilon(s))(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s) j_0^{-1}(z')$$

$$- e^{A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} s} \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} DF_\varepsilon(u_\varepsilon(s))(I + D\Psi_\varepsilon(p_\varepsilon(s)))\Theta_\varepsilon(s)\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E j_0^{-1}(z')ds\Big) = \int_{-\infty}^0 I$$

But, the integrand I can be decomposed, in a similar way as above, as

$$I = \Big(E e^{A_0 \mathbf{Q}_\mathbf{m}^\mathbf{0} s} \mathbf{Q}_\mathbf{m}^\mathbf{0} - e^{A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} s} \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} E \Big) DF_0^\varepsilon(u_0^\varepsilon(s))(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s) j_0^{-1}(z') +$$

$$+ e^{A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} s} \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} \Big(EDF_0^\varepsilon(u_0^\varepsilon) - DF_\varepsilon(Eu_0^\varepsilon(s))E \Big)(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s) j_0^{-1}(z')$$

$$+ e^{A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} s} \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} \Big(DF_\varepsilon(Eu_0^\varepsilon) - DF_\varepsilon(u_\varepsilon(s)) \Big) E(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s)))\Theta_0^\varepsilon(s) j_0^{-1}(z')$$

$$+ e^{A_\varepsilon \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} s} \mathbf{Q}_\mathbf{m}^{\boldsymbol{\varepsilon}} DF_\varepsilon(u_\varepsilon(s)) \Big(E(I + D\Psi_0^\varepsilon(p_0^\varepsilon(s))) - (I + D\Psi_\varepsilon(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} E p_0^\varepsilon(s)))E \Big) \Theta_0^\varepsilon(s) j_0^{-1}(z')$$

$$+e^{A_\varepsilon \mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} s}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} DF_\varepsilon(u_\varepsilon(s))\Big((I+D\Psi_\varepsilon(\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} E p_0^\varepsilon(s))))-(I+D\Psi_\varepsilon(p_\varepsilon(s)))\Big)E\Theta_0^\varepsilon(s)j_0^{-1}(z')$$

$$+e^{A_\varepsilon \mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} s}\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} DF_\varepsilon(u_\varepsilon(s))(I+D\Psi_\varepsilon(p_\varepsilon(s)))\Big(E\Theta_0^\varepsilon(s)-\Theta_\varepsilon(s)\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} E\Big)j_0^{-1}(z')$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

Applying Lemma 2.1.20 and Lemma 2.2.7,

$$\|I_1\|_{X_\varepsilon^\alpha} \leq 2C_5 L_F l_\varepsilon^\alpha(-s) e^{[-2L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon - 1]s}|z'|_{0,\alpha}.$$

Following the same steps as in the proof of Lemma 2.2.10, we obtain,

$$\|I_2\|_{X_\varepsilon^\alpha} \leq 2(\lambda_{m+1}^\varepsilon)^\alpha \beta(\varepsilon) e^{[-2L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon]s}|z'|_{0,\alpha},$$

$$\|I_4\|_{X_\varepsilon^\alpha} \leq (\lambda_{m+1}^\varepsilon)^\alpha L_F \|ED\Psi_0^\varepsilon - D\Psi_\varepsilon E\|_\infty e^{[-2L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon]s}|z'|_{0,\alpha}.$$

For the sake of clarity we will denote by

$$\begin{aligned}\Lambda_3 &= -(2+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - (\theta+1)\lambda_m^\varepsilon - 3\theta\\ \Lambda_4 &= -(4+(\kappa+2)\theta)L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - (\theta+1)\lambda_m^\varepsilon - 3\theta.\end{aligned} \tag{2.2.15}$$

Then, we have,

$$\|I_3\|_{X_\varepsilon^\alpha} \leq 2\kappa(\lambda_{m+1}^\varepsilon)^\alpha LC[\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)]^\theta e^{\Lambda_3 s}|z'|_{0,\alpha},$$

$$\|I_5\|_{X_\varepsilon^\alpha} \leq \kappa L_F(\lambda_{m+1}^\varepsilon)^\alpha MC\left(\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)\right)^\theta e^{\Lambda_3 s}|z'|_{0,\alpha},$$

and for the norm of I_6 we apply Lemma 2.2.10,

$$\|I_6\|_{X_\varepsilon^\alpha} \leq \Big(2(\lambda_{m+1}^\varepsilon)^\alpha L_F C\left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^\theta\right]e^{\Lambda_4 s} +$$

$$(\lambda_{m+1}^\varepsilon)^\alpha L_F\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon E\|_\infty e^{[-4L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon]s}\Big)|z'|_{0,\alpha}.$$

Putting everything together, $\|I\|_{X_\varepsilon^\alpha} \leq \|I_1\|_{X_\varepsilon^\alpha} + \|I_2\|_{X_\varepsilon^\alpha} + \|I_3\|_{X_\varepsilon^\alpha} + \|I_4\|_{X_\varepsilon^\alpha} + \|I_5\|_{X_\varepsilon^\alpha} + \|I_6\|_{X_\varepsilon^\alpha}$, so,

$$\int_{-\infty}^0 \|I\|_{X_\varepsilon^\alpha} ds \leq 2C_5 L_F|z'|_{0,\alpha}\int_{-\infty}^0 l_\varepsilon^\alpha(-s)e^{[-2L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon - 1]s}ds +$$

$$+2(\lambda_{m+1}^\varepsilon)^\alpha\beta(\varepsilon)|z'|_{0,\alpha}\int_{-\infty}^0 e^{[-2L_F(\lambda_m^\varepsilon)^\alpha + \lambda_{m+1}^\varepsilon - \lambda_m^\varepsilon]s}ds +$$

$$+2\kappa(\lambda_{m+1}^\varepsilon)^\alpha LC[\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)]^\theta|z'|_{0,\alpha}\int_{-\infty}^0 e^{\Lambda_3 s}ds +$$

$$+(\lambda_{m+1}^\varepsilon)^\alpha L_F\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon E\|_\infty|z'|_{0,\alpha}\int_{-\infty}^0 e^{[-2L_F(\lambda_m^\varepsilon)^\alpha+\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon]s}ds+$$

$$+\kappa L_F(\lambda_{m+1}^\varepsilon)^\alpha MC\left(\tau(\varepsilon)|\log(\tau(\varepsilon))|+\rho(\varepsilon)\right)^\theta|z'|_{0,\alpha}\int_{-\infty}^0 e^{\Lambda_3 s}ds+$$

$$+2(\lambda_{m+1}^\varepsilon)^\alpha L_F C\left[\beta(\varepsilon)+\left(\tau(\varepsilon)|\log(\tau(\varepsilon))|+\rho(\varepsilon)\right)^\theta\right]|z'|_{0,\alpha}\int_{-\infty}^0 e^{\Lambda_4 s}ds+$$

$$+(\lambda_{m+1}^\varepsilon)^\alpha L_F\|ED\Psi_0^\varepsilon - D\Psi_\varepsilon E\|_\infty|z'|_{0,\alpha}\int_{-\infty}^0 e^{[-4L_F(\lambda_m^\varepsilon)^\alpha+\lambda_{m+1}^\varepsilon-\lambda_m^\varepsilon]s}ds.$$

By Lemma 2.1.15, the gap conditions described in Proposition 2.1.2 and $0<\theta<\tilde{\theta}$, see (2.2.13), for ε small enough,

$$\leq\left(C[\beta(\varepsilon)+(\tau(\varepsilon)|\log(\tau(\varepsilon))|+\rho(\varepsilon))^\theta]+\frac{1}{2}\|ED\Psi_0^\varepsilon-D\Psi_\varepsilon E\|_\infty\right)|z'|_{0,\alpha}$$

Hence,

$$\|[ED\Psi_0^\varepsilon(j_0^{-1}(z))-D\Psi_\varepsilon(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}Ej_0^{-1}(z))\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}E]j_0^{-1}(z')\|_{X_\varepsilon^\alpha}\leq$$

$$\leq\left(C\left[\beta(\varepsilon)+(\tau(\varepsilon)|\log(\tau(\varepsilon))|+\rho(\varepsilon))^\theta\right]+\frac{1}{2}\|ED\Psi_0^\varepsilon-D\Psi_\varepsilon E\|_\infty\right)|z'|_{0,\alpha}.$$

Since Ψ_ε and Ψ_0^ε have bounded support, we consider the sup norm described in (2.2.11) for $u_0\in\mathbf{P}_\mathbf{m}^\mathbf{0}X_0^\alpha$ with $\|u_0\|_{X_0^\alpha}\leq 2\mathcal{R}$, with $\mathcal{R}>0$ an upper bound of the support of all Ψ_ε, $0<\varepsilon\leq\varepsilon_0$, and of Ψ_0^ε.

So,

$$\|ED\Psi_0^\varepsilon-D\Psi_\varepsilon E\|_\infty=$$

$$=\sup_{p\in\mathbf{P}_\mathbf{m}^\mathbf{0}X_0^\alpha,\|p\|_{X_0^\alpha}\leq 2\mathcal{R}}\|ED\Psi_0^\varepsilon(p)-D\Psi_\varepsilon(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}Ep)\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0}X_0^\alpha,X_\varepsilon^\alpha)}$$

$$=\sup_{z\in\mathbb{R}^m,|z|_{0,\alpha}\leq 2\mathcal{R}}\|ED\Psi_0^\varepsilon(j_0^{-1}(z))-D\Psi_\varepsilon(\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}Ej_0^{-1}(z))\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}E\|_{\mathcal{L}(\mathbf{P}_\mathbf{m}^\mathbf{0}X_0^\alpha,X_\varepsilon^\alpha)}\leq$$

$$\leq C\left[\beta(\varepsilon)+(\tau(\varepsilon)|\log(\tau(\varepsilon))|+\rho(\varepsilon))^\theta\right]+\frac{1}{2}\|ED\Psi_0^\varepsilon-D\Psi_\varepsilon E\|_\infty.$$

which implies,

$$\|ED\Psi_0^\varepsilon-D\Psi_\varepsilon E\|_\infty\leq 2C\left[\beta(\varepsilon)+(\tau(\varepsilon)|\log(\tau(\varepsilon))|+\rho(\varepsilon))^\theta\right],$$

with $\theta<\tilde{\theta}$.

Hence, for $\theta<\tilde{\theta}$,

$$\sup_{z\in\mathbb{R}^m}\|ED\Phi_0^\varepsilon(z)-D\Phi_\varepsilon(z)\|_{\mathcal{L}(\mathbb{R}^m,X_\varepsilon^\alpha)}\leq 2C\left[\beta(\varepsilon)+(\tau(\varepsilon)|\log(\tau(\varepsilon))|+\rho(\varepsilon))^\theta\right].$$

Applying Theorem 2.1.4, then

$$\|ED\Phi_0^\varepsilon - D\Phi_\varepsilon\|_{C^1(\mathbb{R}^m,X_\varepsilon^\alpha)} \leq C\left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^\theta\right].$$

Which concludes the proof of the proposition. ■

With this estimate we can analyze in detail the $C^{1,\theta}$-convergence of inertial manifolds for some $\theta < \tilde{\theta}$, small enough. We introduce now the proof of the main result of this subsection.

Proof of Theorem 2.2.2. We want to show the existence of θ^* such that we can prove the convergence of the inertial manifolds Φ_ε to Φ_0^ε, when ε tends to zero in the $C^{1,\theta}$ topology for $\theta < \theta^*$ and obtain a rate of this convergence. That is, an estimate of $\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m,X_\varepsilon^\alpha)}$. Let us choose $\theta^* < \tilde{\theta}$ as close as we want to $\tilde{\theta}$, where $\tilde{\theta}$ is given by (2.2.13), so that Proposition 2.2.12 holds.

As we have mentioned,

$$\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m,X_\varepsilon^\alpha)} = \|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{C^1(\mathbb{R}^m,X_\varepsilon^\alpha)} +$$

$$+ \sup_{z,z'\in\mathbb{R}^m} \frac{\|(D\Phi_\varepsilon - ED\Phi_0^\varepsilon)(z) - (D\Phi_\varepsilon - ED\Phi_0^\varepsilon)(z')\|_{\mathcal{L}(\mathbb{R}^m,X_\varepsilon^\alpha)}}{|z-z'|_{\varepsilon,\alpha}^\theta} =$$

$$= I_1 + I_2.$$

For $\theta < \theta^*$, I_2 can be written as $I_2 = I_{21}\cdot I_{22}$, where

$$I_{21} = \left(\frac{\|(D\Phi_\varepsilon - ED\Phi_0^\varepsilon)(z) - (D\Phi_\varepsilon - ED\Phi_0^\varepsilon)(z')\|_{\mathcal{L}(\mathbb{R}^m,X_\varepsilon^\alpha)}}{|z-z'|_{\varepsilon,\alpha}^{\theta^*}}\right)^{\frac{\theta}{\theta^*}}$$

$$I_{22} = \|(D\Phi_\varepsilon - ED\Phi_0^\varepsilon)(z) - (D\Phi_\varepsilon - ED\Phi_0^\varepsilon)(z')\|_{\mathcal{L}(\mathbb{R}^m,X_\varepsilon^\alpha)}^{1-\frac{\theta}{\theta^*}}$$

Note that, since for each $\varepsilon > 0$, $\Phi_\varepsilon = \Psi_\varepsilon \circ j_\varepsilon^{-1}$, and $\Phi_0^\varepsilon = \Psi_0^\varepsilon \circ j_0^{-1}$ then by the chain rule, for all $z, \bar{v} \in \mathbb{R}^m$,

$$D\Phi_\varepsilon(z)z' = D\Psi_\varepsilon(j_\varepsilon^{-1}(z))(j_\varepsilon^{-1}(z')),$$

$$D\Phi_0^\varepsilon(z)z' = D\Psi_0^\varepsilon(j_0^{-1}(z))(j_0^{-1}(z')).$$

Also, notice that from the definition of j_ε, j_0, we have $j_\varepsilon \circ \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}E = j_0$ or equivalently $j_\varepsilon^{-1} = \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}}E \circ j_0^{-1}$.

Then, applying (2.1.42),

$$I_{21} \leq$$

$$\left(\frac{\|(D\Psi_\varepsilon(j_\varepsilon^{-1}(z)) - D\Psi_\varepsilon(j_\varepsilon^{-1}(z')))j_\varepsilon^{-1} + (ED\Psi_0^\varepsilon(j_0^{-1}(z')) - ED\Psi_0^\varepsilon(j_0^{-1}(z)))j_0^{-1}\|_{\mathcal{L}(\mathbb{R}^m,X_\varepsilon^\alpha)}}{(1-\delta)^{\theta^*}\|j_0^{-1}(z) - j_0^{-1}(z')\|_{X_0^\alpha}^{\theta^*}}\right)^{\frac{\theta}{\theta^*}}$$

Since in the previous subsection we have proved $D\Psi_\varepsilon \in \mathcal{E}_\varepsilon^{\theta,M}$, with $\theta < \theta_0$, in particular we have $D\Psi_\varepsilon \in \mathcal{E}_\varepsilon^{\theta,M}$, with $\theta < \tilde{\theta}$. Without loss of generality we consider $D\Psi_\varepsilon \in \mathcal{E}_\varepsilon^{\theta^*,M}$. Moreover, $\|j_\varepsilon^{-1}\|_{\mathcal{L}(\mathbb{R}^m, \mathbf{P}_\mathbf{m}^\varepsilon X_\varepsilon^\alpha)} = \|\mathbf{P}_\mathbf{m}^\varepsilon E \circ j_0^{-1}\|_{\mathcal{L}(\mathbb{R}^m, \mathbf{P}_\mathbf{m}^\varepsilon X_\varepsilon^\alpha)} \leq \kappa$, see (2.1.11) and (2.1.3). Then, we obtain

$$I_{21} \leq \frac{(M\kappa(\kappa+1))^{\frac{\theta}{\theta^*}} \|j_0^{-1}(z) - j_0^{-1}(z')\|_{X_0^\alpha}^{\theta}}{(1-\delta)^\theta \|j_0^{-1}(z) - j_0^{-1}(z')\|_{X_0^\alpha}^{\theta}} = \frac{(M\kappa(\kappa+1))^{\frac{\theta}{\theta^*}}}{(1-\delta)^\theta}.$$

Note that,

$$I_{22} \leq \left(2\|D\Phi_\varepsilon - ED\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m,\, \mathcal{L}(\mathbb{R}^m, X_\varepsilon^\alpha))}\right)^{1-\frac{\theta}{\theta^*}}.$$

Hence, for $\theta < \theta^*$,

$$\begin{aligned}
&\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m, X_\varepsilon^\alpha)} \leq \\
&\leq \|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} + \|D\Phi_\varepsilon - ED\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m,\, \mathcal{L}(\mathbb{R}^m, X_\varepsilon^\alpha))} + \\
&+ \frac{(M\kappa(\kappa+1))^{\frac{\theta}{\theta^*}}}{(1-\delta)^\theta} \left(2\|D\Phi_\varepsilon - ED\Phi_0^\varepsilon\|_{L^\infty(\mathbb{R}^m,\, \mathcal{L}(\mathbb{R}^m, X_\varepsilon^\alpha))}\right)^{1-\frac{\theta}{\theta^*}}.
\end{aligned}$$

By Theorem 2.1.4 and Proposition 2.2.12, we have

$$\begin{aligned}
&\|\Phi_\varepsilon - E\Phi_0^\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m, X_\varepsilon^\alpha)} \leq \\
&\leq C[\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon)] + 2C\left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^{\theta^*}\right] + \\
&+ \frac{(M\kappa(\kappa+1))^{\frac{\theta}{\theta^*}}}{(1-\delta)^\theta} \left(4C\left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^{\theta^*}\right]\right)^{1-\frac{\theta}{\theta^*}} \leq \\
&\leq \mathbf{C}\left(\left[\beta(\varepsilon) + (\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho(\varepsilon))^{\theta^*}\right]\right)^{1-\frac{\theta}{\theta^*}},
\end{aligned}$$

which shows the result.

■

Chapter 3

A Thin Domain Problem

In this chapter we study the rate of convergence of attractors for a reaction diffusion equation in a thin domain when the thickness ε goes to zero. Our domain is a thin channel obtained by shrinking a fixed domain $Q \subset \mathbb{R}^d$, see Figure 3.1, by a factor ε in (d-1)-directions. The thin channel Q_ε collapses to the one dimensional line segment $[0,1]$ as ε goes to zero.

We consider the following reaction diffusion-equation in Q_ε,

$$\begin{cases} u_t - \Delta u + \alpha u = f(u) & \text{in} \quad Q_\varepsilon, \\ \frac{\partial u}{\partial \nu_\varepsilon} = 0 & \text{in} \quad \partial Q_\varepsilon, \end{cases} \tag{3.0.1}$$

where $\alpha > 0$ is a fixed number, ν_ε the unit outward normal to ∂Q_ε and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear term, with appropriate dissipativity conditions to guarantee the existence of an attractor $\mathcal{A}_\varepsilon \subset H^1(Q_\varepsilon)$.

As the parameter $\varepsilon \to 0$, the thin domain shrinks to the line segment $[0,1]$ and the limiting reaction-diffusion equation is given by

$$\begin{cases} u_t - \frac{1}{g}(g u_x)_x + \alpha u = f(u) & \text{in} \quad (0,1), \\ u_x(0) = u_x(1) = 0. \end{cases} \tag{3.0.2}$$

which also has an attractor $\mathcal{A}_0 \subset H^1(0,1)$.

There are several works in the literature comparing the dynamics of both equations and showing the convergence of $\mathcal{A}_\varepsilon$ to $\mathcal{A}_0$ as $\varepsilon \to 0$, under certain hypotheses. One of the most relevant and pioneer work in this direction is [31], where the authors show that when $d = 2$ and every equilibrium of the limit problem (3.1.7) is hyperbolic, then the attractors behave continuously and moreover, the flow in the attractors of both systems are topologically conjugate. In order to accomplish this task, the authors exploit the fact that the limit problem is one dimensional, which allows them to construct inertial manifolds for (3.1.3) and (3.1.7) which will be close in the C^1 topology. Restricting the flow to these inertial manifolds, and using that the limit problem is Morse-Smale (under the condition that all equilibria being hyperbolic, see [33]) they prove the C^0-conjugacy of the flows. Moreover the method of constructing the inertial manifolds for fixed $\varepsilon \in [0, \varepsilon_0]$ consists in using the method described in [38]. They consider the finite dimensional linear manifold given by the span of the eigenfunctions corresponding to the first m eigenvalues of the elliptic operator and let evolve this linear manifold with the nonlinear flow, which ω-limit set is a C^1 manifold and it is the inertial manifold, which, as a matter of fact it is a graph over the finite dimensional linear manifold. This method provides them with an estimate of the distance of the inertial manifolds of the order of ε^γ for some $\gamma < 1$. Later on, reducing the system to the inertial manifolds and using the general techniques to estimate the distance of attractors for gradient flows, see [30] Theorem 2.5, give them the estimate $\varepsilon^{\gamma'}$ with some $\gamma' < \gamma < 1$ which depends on the number of equilibria of the limit problem and other characteristics of the problem.

Our setting is more general than the one from [31], since we consider general $d-$dimensional thin domains (not just 2-dimensional). Moreover, our approach to this problem has some differences with respect to theirs. In our case, we will also construct inertial manifolds, but we will construct them following the Lyapunov-Perron method, as developed in Chapter 2. This method, as it is shown in the previous chapter, provides us with a good estimate of the C^0 distance of the inertial manifolds (which is of order $\varepsilon|\ln(\varepsilon)|$) and with the $C^{1,\theta}$ convergence of this manifolds. Moreover, we have to construct the inertial manifolds as graphs of functions in a fractional power space X^α_ε with some $0 < \alpha < 1/2$. Moreover, as it will become clear in the chapter, we cannot take $\alpha = 0$, since we will not be able to show smoothness of the inertial manifolds, nor $\alpha = 1/2$ since for this value of α we cannot prove that the appropriate gap condition on the spectrum, which is a necessary condition for the construction of the inertial manifolds. Hence we are forced to work in the family of spaces X^α_ε with some $0 < \alpha < 1/2$ for which we need to know some "uniform Sobolev embeddings".

Once the Inertial Manifolds are constructed and we have a good estimate of its distance we can project the systems to these inertial manifolds and obtain the reduced systems, which are finite dimensional. The limit reduced system will be a Morse-Smale gradient like system, as defined in Chapter 1. Then the shadowing theory developed in Chapter 1 plays an important role in obtaining the rates of convergence of the attractors.

Let us mention that the estimate we find on the Hausdorff symmetric distance of the attractors is the following (see Theorem 3.1.2),

$$\mathrm{dist}_{H^1(Q_\varepsilon)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq C\varepsilon^{\frac{d+1}{2}} |\log(\varepsilon)|$$

which improves the one obtained in [31].

We describe now the contents of this chapter:

In section 3.1 we give a complete description of the thin domain Q_ε, will set up the basic notation we will need. We also introduce the main result of the paper.

In section 3.2 we study the related elliptic problem, obtaining an estimate for the distance of the resolvent operators and proving this estimate is optimal.

In section 3.3 we analyze the nonlinearity and we prepare it to the construction of inertial manifolds. We make an appropriate cut off of the non-linear term and analyze the conditions this new nonlinearity satisfies.

In section 3.4 we construct the corresponding inertial manifolds, reducing our problem to a finite dimensional one.

In section 3.5 using the optimal estimate proved in section 3.2 and the shadowing result obtained in Chapter 1 we provide an almost optimal rate of convergence of attractors.

And, at the end, we present an appendix, section 3.6, which describes the needed relation between fractional power spaces and interpolation spaces. We show an uniform equivalence in ε between them.

3.1. Setting of the problem and main results

In this section we set up the problem, describing clearly the domain and the equations we are dealing with. We will also state our main result of the distance of attractors. We end up the section with some notation and technical results needed thereafter.

We start describing the thin domain. Let $\Omega = (0,1)$ and let Q be the set

$$Q = \{(x, \mathbf{y}) \in \mathbb{R}^d : 0 \leq x \leq 1, \;\; \mathbf{y} \in \Gamma_x^1\},$$

with $d \geq 2$, and Γ_x^1 diffeomorphic to the unit ball in $\mathbb{R}^{d-1}$, $B(0,1)$, for all $x \in [0,1]$, see Figure 3.1, that is, we assume that for each $x \in [0,1]$, there exists a C^1 dipheomorphism $\mathbf{L}_x$

$$\mathbf{L}_x : B(0,1) \longrightarrow \Gamma_x^1 \subset \mathbb{R}^{d-1}. \tag{3.1.1}$$

We also assume that, if we define

$$\begin{cases} \mathbf{L} : (0,1) \times B(0,1) & \longrightarrow & Q \\ \qquad\qquad (x, \mathbf{y}) & \mapsto & (x, \mathbf{L}_x(\mathbf{y})) \end{cases} \tag{3.1.2}$$

then $\mathbf{L}$ is a C^1 diffeomorphism. The boundary of Q has two distinguished parts, the one formed by $\Gamma_0^1 \cup \Gamma_1^1$ (the two lids of the thin domain) and the lateral boundary $\partial_l Q = \{(x,y) : x \in (0,1), y \in \partial\Gamma_x^1\}$

Our thin channel, or thin domain will be defined by

$$Q_\varepsilon = \{(x, \varepsilon\mathbf{y}) \in \mathbb{R}^d : (x, \mathbf{y}) \in Q\}, \qquad \varepsilon \in (0,1).$$

Notice that this set is obtained by shrinking the set Q by a factor ε in the $(d-1)$-directions given by the variable $\mathbf{y} \in \mathbb{R}^{d-1}$. This domain gets thinner and thinner as $\varepsilon \to 0$ and it approaches the one dimensional line segment given by $\Omega \times \{\mathbf{0}\} = (0,1) \times \{\mathbf{0}\}$.

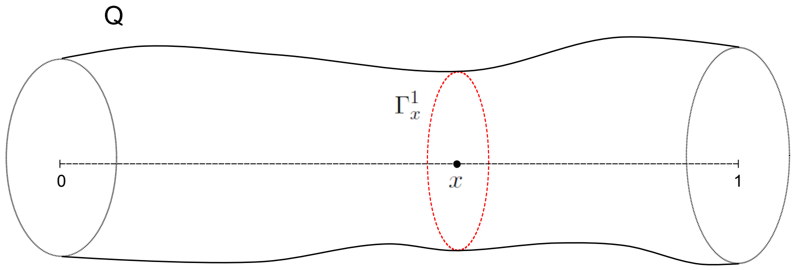

Figure 3.1: Domain Q with $d = 3$.

We denote by $g(x) := |\Gamma_x^1|$ the $(d-1)$-dimensional Lebesgue measure of the set Γ_x^1. From the hypothesis of the smoothness of the map $\mathbf{L}$ above, see (3.1.2), we have that g is a smooth function defined in $[0, 1]$. In particular, there exist $g_0, g_1 > 0$ such that $g_0 \leq g(x) \leq g_1$ for all $x \in [0, 1]$.

Remark 3.1.1. *An important subclass of these thin domains are those whose transversal sections Γ_x^1 are disks centered at the origin of radius $r(x)$, that is,*

$$Q = \{(x, \mathbf{y}) \in \mathbb{R}^d : 0 \leq x \leq 1, |\mathbf{y}| < r(x)\}.$$

In this particular case, $g(x) = |B(0, 1)|r(x)^{d-1}$, with $|B(0, 1)|$ the Lebesgue measure of the unit ball in $\mathbb{R}^m$. The diffeomorphism $\mathbf{L}$ defined in (3.1.2) is given by,

$$\mathbf{L}(x, \mathbf{y}) = (x, r(x)\mathbf{y}).$$

We consider the following reaction-diffusion equation in Q_ε, $0 < \varepsilon \leq \varepsilon_0$,

$$\begin{cases} u_t - \Delta u + \alpha u = f(u) & \text{in} \quad Q_\varepsilon, \\ \frac{\partial u}{\partial \nu_\varepsilon} = 0 & \text{in} \quad \partial Q_\varepsilon, \end{cases} \tag{3.1.3}$$

where $\alpha > 0$ is a fixed number, ν_ε the unit outward normal to ∂Q_ε and $f : \mathbb{R} \to \mathbb{R}$ a C^2-function satisfying the following growth condition

$$|f'(s)| \leq C(1 + |s|^{\rho-1}), \quad s \in \mathbb{R} \tag{3.1.4}$$

for some $\rho \geq 1$, and the dissipative condition,

$$\exists\, M > 0, \quad \text{s.t.} \quad f(s) \cdot s \leq 0, \qquad |s| \geq M. \tag{3.1.5}$$

With the growth condition (3.1.4) we know that problem (3.1.3) is locally well posed in some functional space of the type $L^r(Q_\varepsilon)$ for some $r > 1$, maybe large enough, or $W^{1,p}(Q_\varepsilon)$, see [9]. With the dissipative condition and with some regularity arguments, see [9], we obtain that solutions are globally defined and with the aid of the maximum principle there exist uniform asymptotic bounds in the sup norm of the solutions. That is, for any initial condition ϕ_ε there exists a time τ, that may depend on ε and on the initial condition, such that the solution starting at ϕ_ε after time τ is uniformly bounded by M, that is $|u(t,x,\phi_\varepsilon)| \le M$ for $t \ge \tau$, with M from (3.1.5). This uniform asymptotic bounds together with parabolic regularity theory imply that the equation (3.1.3) has an attractor $\mathcal{A}_\varepsilon \subset H^1(Q_\varepsilon) \cap L^\infty(Q_\varepsilon)$ satisfying the uniform bound

$$\|u_\varepsilon\|_{L^\infty(Q_\varepsilon)} \le M, \qquad \text{for all } u_\varepsilon \in \mathcal{A}_\varepsilon \tag{3.1.6}$$

The limit problem of (3.1.3) is given by, see [31],

$$\begin{cases} u_t - \frac{1}{g}(gu_x)_x + \alpha u = f(u) & \text{in} \quad (0,1), \\ \quad u_x(0) = u_x(1) = 0. \end{cases} \tag{3.1.7}$$

and, just as the analysis above, this equation has also an attractor $\mathcal{A}_0 \subset H^1(0,1) \cap L^\infty(0,1)$ satisfying also the bounds

$$\|u_0\|_{L^\infty(0,1)} \le M. \tag{3.1.8}$$

Observe that the dynamical system generated by this equation has a gradient structure (see [25]) and in particular its attractor is formed by equilibria and connections among them. Moreover, if all equilibria are hyperbolic then we have only a finite number of them and the system has a Morse-Smale structure (see [33]).

Notice that in a natural way we may consider the attractor $\mathcal{A}_0$ as a subset of $H^1(Q_\varepsilon)$, just by considering that any function $u_0(x)$ defined in $(0,1)$ is extended to all of Q_ε by $\tilde{u}_0(x,\mathbf{y}) = u_0(x)$.

We now introduce the main result of the paper.

Theorem 3.1.2. *Under the notations above and assuming that all equilibria of problem* (3.1.7) *are hyperbolic, we have*

$$dist_{H^1(Q_\varepsilon)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) \le C\varepsilon^{\frac{d+1}{2}} |\log(\varepsilon)|, \tag{3.1.9}$$

with $dist_X(\cdot,\cdot)$ the symmetric Haussdorf distance in the space X.

Recall that $\mathrm{dist}_X(\cdot,\cdot)$ is defined in (0.4.2).

To prove this result, we show the existence of inertial manifolds, invariant and exponential attracting finite dimensional manifolds, which contain the attractors $\mathcal{A}_\varepsilon$, $\varepsilon \ge 0$ and reduce our infinite dimensional dynamical system we want to study to a finite dimensional one. Then we will have finite dimensional system which, via an isomorphism, will be transformed in ordinary differential equations in $\mathbb{R}^m$. We prove these inertial manifolds are smooth enough. Then, with the shadowing techniques from Chapter 1 we will be able to obtain an estimate for the distance of the associated attractors in $\mathbb{R}^m$ of order the distance of the time one maps related to the ordinary differential equations. Moreover, applying results from Chapter 2, we estimate the distance of these inertial manifolds by obtaining an optimal estimate for the distance of the related resolvent operators. These two estimates, the distance of the inertial manifolds and the distance of the associated attractors in $\mathbb{R}^m$, will allow us to give a rate for the distance of attractors $\mathcal{A}_0$ and $\mathcal{A}_\varepsilon$ of order $\varepsilon^{\frac{d+1}{2}}|\log(\varepsilon)|$. This rate appears to be optimal, apart from the $|\log(\varepsilon)|$ factor.

Next, we present the notation and some conditions needed for the proof.

As we have noted above, the attractors of both equations (3.1.3) and (3.1.7) have uniform L^∞ bounds, as expressed in (3.1.6) and (3.1.8). This fact will allow us to cut off the nonlinearity f outside the interval $(-M, M)$ so that the new nonlinearity that we will still denote by f has compact support and coincides with the old one in $(-M, M)$, satisfies

$$|f(s)| + |f'(s)| + |f''(s)| \leq L_f \qquad \text{for all} \quad s \in \mathbb{R}, \tag{3.1.10}$$

and the dissipative condition (3.1.5) still holds for the new f. Moreover, since the attractors for the old nonlinearity satisfy (3.1.6) and (3.1.8) and the new f coincides with the old one in $(-M, M)$, then the attractors for the new equations are exactly the same as the attractors for the original equations. This means that we may assume from the beginning that the nonlinearity f satisfies (3.1.10)

When dealing with problems where the domain varies it is sometimes convenient to make transformations, as simple as possible, so that we transform all problems to a fixed reference domain. This will imply in many instances that the parameter appears in the equation and usually it will show up as a singular parameter. In our case, we will transform problem (3.1.3) in a problem in the fixed set $Q = \{(x, \mathbf{y}) \in \mathbb{R}^d : 0 \leq x \leq 1, \mathbf{y} \in \Gamma_x^1\}$, (Figure 3.1). The transformation we will use is $(x, \mathbf{y}) \to (x, \frac{\mathbf{y}}{\varepsilon})$. With this transformation, the reaction-diffusion equation (3.1.3) is transformed into the following equation on the fixed domain Q,

$$\left\{ \begin{array}{rll} u_t - \frac{\partial^2 u}{\partial x^2} - \frac{1}{\varepsilon^2}\Delta_{\mathbf{y}} u + \alpha u = f(u) & \text{in} & Q, \\ \frac{\partial u}{\partial \nu_x} + \frac{1}{\varepsilon^2}\frac{\partial u}{\partial \nu_{\mathbf{y}}} = 0 & \text{on} & \partial Q \end{array} \right. \tag{3.1.11}$$

where $\nu = (\frac{\partial u}{\partial \nu_x}, \frac{\partial u}{\partial \nu_{\mathbf{y}}})$ is the unit outward normal to ∂Q.

The natural spaces to analyze (3.1.11) are given by,

$$H^1_{\boldsymbol{\varepsilon}}(Q) := (H^1(Q), \|\cdot\|_{H^1_{\boldsymbol{\varepsilon}}(Q)}),$$

with the norm

$$\|u\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} := \left(\int_Q (|\nabla_x u|^2 + \frac{1}{\varepsilon^2}|\nabla_{\mathbf{y}} u|^2 + |u|^2) dx d\mathbf{y} \right)^{1/2},$$

and $L^2(Q)$ with the usual norm $\|\cdot\|_{L^2(Q)}$.

Notice that if we define the isomorphism $\mathbf{i}_{\boldsymbol{\varepsilon}} : L^2(Q_\varepsilon) \to L^2(Q)$ as

$$\mathbf{i}_{\boldsymbol{\varepsilon}}(u)(x, \mathbf{y}) := u(x, \varepsilon \mathbf{y}),$$

its restriction to $H^1(Q_\varepsilon)$ is also an isomorphism from $H^1(Q_\varepsilon)$ to $H^1(Q)$ (or equivalently to $H^1_{\boldsymbol{\varepsilon}}(Q)$). Then we easily have the following identities:

$$\|\mathbf{i}_{\boldsymbol{\varepsilon}}(u)\|_{L^2(Q)} = \varepsilon^{-\frac{d-1}{2}}\|u\|_{L^2(Q_\varepsilon)} \tag{3.1.12}$$

$$\|\mathbf{i}_{\boldsymbol{\varepsilon}}(u)\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} = \varepsilon^{-\frac{d-1}{2}}\|u\|_{H^1(Q_\varepsilon)} \tag{3.1.13}$$

The isomorphism $\mathbf{i}_{\boldsymbol{\varepsilon}}$ also allows us to relate easily the semigroups generated by (3.1.3) and (3.1.11) as follows: if $S_\varepsilon(t)$ is the semigroup generated by (3.1.3) and $\tilde{S}_\varepsilon(t)$ the one from (3.1.11), then we have

$$S_\varepsilon(t)(\cdot) := \mathbf{i}_{\boldsymbol{\varepsilon}}^{-1} \circ \tilde{S}_\varepsilon(t) \circ \mathbf{i}_{\boldsymbol{\varepsilon}}(\cdot),$$

The limit problem of (3.1.11) is also given by (3.1.7).

The natural spaces to treat the limit problem are the following

$$L^2_g(0,1) := (L^2(0,1), \|\cdot\|_{L^2_g(0,1)}) \qquad \text{with} \quad \|u\|_{L^2_g(0,1)} := \left(\int_0^1 g(x)|u(x)|^2dx\right)^{\frac{1}{2}},$$

and

$$H^1_g(0,1) := (H^1(0,1), \|\cdot\|_{H^1_g(0,1)}) \qquad \text{with} \quad \|u\|_{H^1_g(0,1)} := \left(\int_0^1 g(x)(|u_x|^2+|u|^2)dx\right)^{\frac{1}{2}}.$$

Throughout this paper we will denote by $|\cdot|$ the norm in $\mathbb{R}^d$.

We define an extension operator which maps functions defined in $[0,1]$ into functions defined in Q. The natural way to construct this operator is to extend the functions defined in $[0,1]$ constantly in the other $d-1$ variables. Therefore we denote by E the transformation,

$$\begin{array}{rcl} E : L^2_g(0,1) & \longrightarrow & L^2(Q) \\ u & \mapsto & E(u)(x,\mathbf{y}) = u(x) \end{array} \tag{3.1.14}$$

In a similar fashion we may define the transformation $E_\varepsilon : L^2_g(0,1) \to L^2(Q_\varepsilon)$ defined as $(E_\varepsilon u)(x,\mathbf{y}) = u(x)$. The difference with E is that E_ε lands in $L^2(Q_\varepsilon)$. As a mather of fact, $E_\varepsilon = \mathbf{i}_{\boldsymbol{\varepsilon}}^{-1} \circ E$.

These transformations can also be considered as $E : H^1_g(0,1) \longrightarrow H^1_{\boldsymbol{\varepsilon}}(Q)$ and $E_\varepsilon : H^1_g(0,1) \to H^1(Q_\varepsilon)$. Moreover, if for $0 < \alpha < \frac{1}{2}$ we denote by X^α_ε, $\varepsilon \geq 0$, the fractional power spaces, see [32], related to the elliptic part of (3.1.7) and (3.1.11), these transformations can be considered too as $E : X^\alpha_0 \longrightarrow X^\alpha_\varepsilon$.

To compare functions from $L^2(Q)$ and $L^2_g(0,1)$ (and from X^α_ε and X^α_0, respectively) we also need a projection operator M, defined as follows,

$$\begin{array}{rcl} M : L^2(Q) & \longrightarrow & L^2_g(0,1) \\ u & \mapsto & M(u)(x) = \frac{1}{|\Gamma^1_x|}\int_{\Gamma^1_x} u(x,\mathbf{y})d\mathbf{y}, \end{array} \tag{3.1.15}$$

similary, we may define the map,

$$\begin{array}{rcl} M_\varepsilon : L^2(Q_\varepsilon) & \longrightarrow & L^2_g(0,1) \\ u & \mapsto & M_\varepsilon(u)(x) = \frac{1}{|\Gamma^\varepsilon_x|}\int_{\Gamma^\varepsilon_x} u(x,\mathbf{y})d\mathbf{y}, \end{array} \tag{3.1.16}$$

and, in the same way, for $0 < \alpha < \frac{1}{2}$, $M : X^\alpha_\varepsilon \longrightarrow X^\alpha_0$. Moreover $M : H^1_{\boldsymbol{\varepsilon}}(Q) \longrightarrow H^1_g(0,1)$ and $M_\varepsilon : H^1(Q_\varepsilon) \longrightarrow H^1_g(0,1)$.

With respect to the extension and projection operators defined above, we can show,

Lemma 3.1.3. *We have the following*
i) The projection operators M and M_ε are bounded with norm

$$\|M\|_{\mathcal{L}(L^2(Q),L^2_g(0,1))} \leq 1, \quad \|M\|_{\mathcal{L}(H^1_{\boldsymbol{\varepsilon}}(Q),H^1_g(0,1))} \leq 1, \quad \|M_\varepsilon\|_{\mathcal{L}(L^2(Q_\varepsilon),L^2_g(0,1))} \leq \varepsilon^{\frac{1-d}{2}}.$$

ii) The extension operators E and E_ε satisfy

$$\|Eu\|_{L^2(Q)} = \|u\|_{L^2_g(0,1)}, \quad \|E_\varepsilon u\|_{L^2(Q_\varepsilon)} = \varepsilon^{\frac{d-1}{2}}\|u\|_{L^2_g(0,1)} \quad \forall u \in L^2_g(0,1)$$

$$\|Eu\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} = \|u\|_{H^1_g(0,1)}, \quad \forall u \in H^1_g(0,1)$$

iii) There exists a constant $\beta > 0$ such that

$$\|u_\varepsilon - EMu_\varepsilon\|^2_{L^2(Q)} \leq \beta\|\nabla_{\mathbf{y}} u_\varepsilon\|^2_{L^2(Q)}, \quad \forall u_\varepsilon \in H^1(Q)$$

$$\|w_\varepsilon - E_\varepsilon M_\varepsilon w_\varepsilon\|^2_{L^2(Q_\varepsilon)} \leq \beta\varepsilon^2\|\nabla_{\mathbf{y}} w_\varepsilon\|^2_{L^2(Q_\varepsilon)}, \quad \forall w_\varepsilon \in H^1(Q_\varepsilon)$$

iv) Let $K \subset X^\alpha_0$ a compact set. Then,

$$\sup_{u_0\in K}\left|\|Eu_0\|_{X^\alpha_\varepsilon} - \|u_0\|_{X^\alpha_0}\right| \to 0, \qquad \text{as } \varepsilon \to 0.$$

Proof. The proof of i) and ii) are straightforward. For instance, if $u \in L^2(Q)$ and $(x,\mathbf{y}) \in Q$, then,

$$\|Mu\|_{L^2_g(0,1)} = \left(\int_0^1 g(x)|Mu(x)|^2dx\right)^{\frac{1}{2}} = \left(\int_0^1 g(x)\left|\frac{1}{|\Gamma^1_x|}\int_{\Gamma^1_x} u(x,\mathbf{y})d\mathbf{y}\right|^2 dx\right)^{\frac{1}{2}}.$$

Remember $|\Gamma^1_x| = g(x)$, so,

$$= \left(\int_0^1 g(x)^{-1}\left|\int_{\Gamma^1_x} u(x,\mathbf{y})d\mathbf{y}\right|^2 dx\right)^{\frac{1}{2}} \overset{\text{Hölder ineq.}}{\leq} \left(\int_0^1 g(x)^{-1}|\Gamma^1_x|\int_{\Gamma^1_x}|u(x,\mathbf{y})|^2d\mathbf{y}dx\right)^{\frac{1}{2}} =$$

$$= \left(\int_0^1\int_{\Gamma^1_x}|u(x,\mathbf{y})|^2d\mathbf{y}dx\right)^{\frac{1}{2}} = \|u\|_{L^2(Q)}.$$

The equality holds if u is independent of $\mathbf{y}$ in Q. The other statements are obtained in a similar way.
iii) Observe that,

$$\|u_\varepsilon - EMu_\varepsilon\|^2_{L^2(Q)} = \int_0^1 \int_{\Gamma^1_x} |u_\varepsilon(x,\mathbf{y}) - (Mu_\varepsilon)(x)|^2 d\mathbf{y}dx.$$

But, by Poincare inequality

$$\int_{\Gamma^1_x} |u_\varepsilon(x,\mathbf{y}) - (Mu_\varepsilon)(x)|^2 d\mathbf{y} \leq \frac{1}{\lambda_2(\Gamma^1_x)} \int_{\Gamma^1_x} |\nabla_{\mathbf{y}} u_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y},$$

where $\lambda_2(\Gamma^1_x)$ is the second Neumann eigenvalue in Γ^1_x.

Let us see that there exists a $\hat{\lambda}_2 > 0$ such that,

$$\lambda_2(\Gamma^1_x) \geq \hat{\lambda}_2 > 0, \qquad \forall x \in [0,1].$$

If this is not the case, then there exists a sequence $x_n \to x_0 \in [0,1]$ such that $\lambda_2(\Gamma^1_{x_n}) \to 0$ as $n \to \infty$. But $\Gamma^1_{x_n}$ for n large enough is C^1 close to $\Gamma^1_{x_0}$ and therefore, by the continuity of the Neumann eigenvalues under C^1- perturbations, see [4], we have that $\lambda_2(\Gamma^1_{x_0}) = 0$. But this means that $\Gamma^1_{x_0}$ is not a connected domain and therefore $\Gamma^1_{x_0}$ is not diffeomorphic to the unit ball $B(0,1)$.

Hence,

$$\int_{\Gamma^1_x} |u_\varepsilon(x,\mathbf{y}) - (Mu_\varepsilon)(x)|^2 d\mathbf{y} \leq \frac{1}{\hat{\lambda}_2} \int_{\Gamma^1_x} |\nabla_{\mathbf{y}} u_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y}, \qquad \forall x \in [0,1].$$

Then,

$$\int_Q |u_\varepsilon(x,\mathbf{y}) - (Mu_\varepsilon)(x)|^2 d\mathbf{y}dx \leq \frac{1}{\hat{\lambda}_2} \int_0^1 \int_{\Gamma^1_x} |\nabla_{\mathbf{y}} u_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y}dx =$$

$$= \beta \int_Q |\nabla_{\mathbf{y}} u_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y}dx,$$

with $\beta = \frac{1}{\hat{\lambda}_2}$.

For the inequality in the domain Q_ε, note again that, for $w_\varepsilon \in H^1(Q_\varepsilon)$,

$$\|w_\varepsilon - E_\varepsilon M_\varepsilon w_\varepsilon\|^2_{L^2(Q_\varepsilon)} = \int_0^1 \int_{\Gamma^\varepsilon_x} |w_\varepsilon(x,\mathbf{y}) - (M_\varepsilon w_\varepsilon)(x)|^2 d\mathbf{y}dx.$$

By the Poincare's inequality, we have,

$$\int_{\Gamma^\varepsilon_x} |w_\varepsilon(x,\mathbf{y}) - (M_\varepsilon w_\varepsilon)(x)|^2 d\mathbf{y} \leq \frac{1}{\lambda_2(\Gamma^\varepsilon_x)} \int_{\Gamma^\varepsilon_x} |\nabla_{\mathbf{y}} u_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y} =$$

$$= \varepsilon^2 \frac{1}{\lambda_2(\Gamma^1_x)} \int_{\Gamma^\varepsilon_x} |\nabla_{\mathbf{y}} u_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y}.$$

We have seen above that there exists a $\hat{\lambda}_2 > 0$ such that $\lambda_2(\Gamma_x^1) \geq \hat{\lambda}_2 > 0$ for all $x \in [0,1]$. Then,

$$\int_{Q_\varepsilon} |w_\varepsilon(x,\mathbf{y}) - (M_\varepsilon w_\varepsilon)(x)|^2 d\mathbf{y}dx \leq \varepsilon^2 \frac{1}{\hat{\lambda}_2} \int_0^1 \int_{\Gamma_x^\varepsilon} |\nabla_{\mathbf{y}} w_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y}dx =$$

$$= \varepsilon^2 \beta \int_{Q_\varepsilon} |\nabla_{\mathbf{y}} w_\varepsilon(x,\mathbf{y})|^2 d\mathbf{y}dx.$$

iv) Since $K \subset X_0^\alpha$ is a compact set, for $\eta > 0$ there exist $u_0^1, ..., u_0^{k(\eta)} \in K$ such that

$$K \subset \bigcup_{i=1}^{k(\eta)} B(u_0^i, \eta).$$

Then, for each $u_0 \in K$, there exists $i \in \{1, 2, ..., k(\eta)\}$, such that $\|u_0^i - u_0\|_{X_0^\alpha} \leq \eta$.

Moreover, by continuity of eigenvalues, see Chapter 2, we have for each $i \in \{1, 2, ..., k(\eta)\}$

$$\|Eu_0^i\|_{X_\varepsilon^\alpha} \to \|u_0^i\|_{X_0^\alpha}. \tag{3.1.17}$$

If we write $\|Eu_0\|_{X_\varepsilon^\alpha} = \|E(u_0 - u_0^i) + Eu_0^i\|_{X_\varepsilon^\alpha}$, then,

$$\|Eu_0\|_{X_\varepsilon^\alpha} = \|E(u_0 - u_0^i) + Eu_0^i\|_{X_\varepsilon^\alpha} \leq 2e^2\eta + \|Eu_0^i\|_{X_\varepsilon^\alpha}.$$

Hence,

$$\big|\|Eu_0\|_{X_\varepsilon^\alpha} - \|Eu_0^i\|_{X_\varepsilon^\alpha}\big| \leq 2e^2\eta.$$

From (3.1.17), we know that there exists an $\varepsilon(\eta)$ such that, for $0 \leq \varepsilon \leq \varepsilon(\eta)$,

$$\big|\|Eu_0^i\|_{X_\varepsilon^\alpha} - \|u_0^i\|_{X_0^\alpha}\big| \leq \eta,$$

and,

$$\big|\|Eu_0\|_{X_\varepsilon^\alpha} - \|u_0^i\|_{X_0^\alpha}\big| = \big|\|\|Eu_0\|_{X_\varepsilon^\alpha} - \|Eu_0^i\|_{X_\varepsilon^\alpha} + \|Eu_0^i\|_{X_\varepsilon^\alpha} - \|u_0^i\|_{X_0^\alpha}\big| \leq 2e^2\eta + \eta,$$

for $0 \leq \varepsilon \leq \varepsilon(\eta)$. So,

$$\big|\|Eu_0\|_{X_\varepsilon^\alpha} - \|u_0\|_{X_0^\alpha}\big| =$$

$$\big|\|Eu_0\|_{X_\varepsilon^\alpha} - \|Eu_0^i\|_{X_\varepsilon^\alpha} + \|Eu_0^i\|_{X_\varepsilon^\alpha} - \|u_0^i\|_{X_0^\alpha} + \|u_0^i\|_{X_0^\alpha} - \|u_0\|_{X_0^\alpha}\big| \leq (2e^2 + 2)\eta,$$

for $0 < \varepsilon \leq \varepsilon(\eta)$.

That is, for any $K \subset X_0^\alpha$ and K a compact set,

$$\sup_{u_0 \in K} \big|\|Eu_0\|_{X_\varepsilon^\alpha} - \|u_0\|_{X_0^\alpha}\big| \to 0, \qquad \text{as} \quad \varepsilon \to 0.$$

■

Remark 3.1.4. *Since, as we have just proved,*

$$\|M\|_{\mathcal{L}(L^2(Q),L^2_g(0,1))} \le 1, \qquad \textit{and} \qquad \|M\|_{\mathcal{L}(H^1_{\boldsymbol{\varepsilon}}(Q),H^1_g(0,1))} \le 1,$$

then, by interpolation theory, see [55], for $0 < 2\alpha < 1$*, and for any* $u \in [L^2(Q), H^1_{\boldsymbol{\varepsilon}}(Q)]_{2\alpha}$ *with* $\|u\|_{[L^2(Q),H^1_{\boldsymbol{\varepsilon}}(Q)]_{2\alpha}} = 1$*,*

$$\|Mu\|_{[L^2_g(0,1),H^1_g(0,1)]_{2\alpha}} \le \|Mu\|^{2\alpha}_{L^2_g(0,1)} \|Mu\|^{1-2\alpha}_{H^1_g(0,1)} \le 1.$$

We know that for $0 < 2\alpha < 1$*, see the appendix of this chapter, (Section 3.6),*

$$X^\alpha_0 = [L^2_g(0,1), H^1_g(0,1)]_{2\alpha},$$

and the norms are uniformly equivalent in ε *, see 3.6.1. Then, the norm* $\|M\|_{\mathcal{L}(X^\alpha_\varepsilon,X^\alpha_0)}$*, is uniformly bounded. More precisely, in [51], checking with detail the proof of Theorem 3 we obtain*

$$\|Mu\|_{X^\alpha_0} \le 2e^2 \|u\|_{X^\alpha_\varepsilon}. \tag{3.1.18}$$

For the operator $E : X^\alpha_0 \longrightarrow X^\alpha_\varepsilon$ *we obtain too,*

$$\|Eu\|_{X^\alpha_\varepsilon} \le 2e^2 \|u\|_{X^\alpha_0}, \qquad \forall u \in X^\alpha_0, \tag{3.1.19}$$

applying exactly the same arguments.

Note that operators E, M satisfy the hypothesis (2.1.3) of Chapter 2 with $\kappa = 2e^2$.

3.2. Estimates of the elliptic part

As we mentioned in the introduction, a very important ingredient in comparing the dynamics of both problems is the convergence of the resolvent operators. In this section we will obtain rates of the convergence of these resolvents, proving in particular that hypothesis **(H1)** from Chapter 2 holds and obtaining an estimate of $\tau(\varepsilon)$, see (2.1.7), of the order of ε. We will also show that these estimates are optimal.

We consider the elliptic problems,

$$\left\{ \begin{array}{lll} -\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{1}{\varepsilon^2}\Delta_y u_\varepsilon + \alpha u_\varepsilon & = f_\varepsilon, & \text{in} \quad Q \\ \frac{\partial u}{\partial \nu_x} + \frac{1}{\varepsilon^2}\frac{\partial u}{\partial \nu_{\mathbf{y}}} & = 0 & \text{on} \quad \partial Q, \end{array} \right. \tag{3.2.1}$$

and

$$\left\{ \begin{array}{lll} -\frac{1}{g}(g v_{\varepsilon x})_x + \alpha v_\varepsilon & = h_\varepsilon, & \text{in} \quad (0,1) \\ v_{\varepsilon_x}(0) = v_{\varepsilon_x}(1) & = 0, & \end{array} \right. \tag{3.2.2}$$

with $f_\varepsilon \in L^2(Q)$, $u_\varepsilon \in H^1_{\boldsymbol{\varepsilon}}(Q)$ and $h_\varepsilon \in L^2_g(0,1)$, $v_\varepsilon \in H^1_g(0,1)$. Notice that the existence and uniqueness of solutions of the problems above is guaranteed by Lax-Milgram theorem.

We can prove the following proposition.

Proposition 3.2.1. *Let $f_\varepsilon \in L^2(Q)$ and let $h_\varepsilon = Mf_\varepsilon$. We define the functions $u_\varepsilon \in H^1_{\boldsymbol{\varepsilon}}(Q)$ and $v_\varepsilon \in H^1_g(0,1)$ as the solutions of the linear problems (3.2.1) and (3.2.2), respectively. Then, there exist a constant $C > 0$ independent of ε and f_ε such that,*

$$\|u_\varepsilon - Ev_\varepsilon\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} \le C\varepsilon \|f_\varepsilon\|_{L^2(Q)}.$$

Proof. The proof of this result follows similar ideas as the proof of Proposition A.8 from [6].

Remember that

$$Q_\varepsilon = \{(x, \varepsilon\mathbf{y}) \in \mathbb{R}^d : (x, \mathbf{y}) \in Q\},$$

where

$$Q = \{(x, \mathbf{y}) \in \mathbb{R}^d : 0 \le x \le 1, \ \ \mathbf{y} \in \Gamma^1_x\},$$

and

$$H^1_{\boldsymbol{\varepsilon}}(Q) := (H^1(Q), \| \cdot \|_{H^1_{\boldsymbol{\varepsilon}}(Q)}),$$

with the norm

$$\|u\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} := \left(\int_Q (|\nabla_x u|^2 + \frac{1}{\varepsilon^2} |\nabla_{\mathbf{y}} u|^2 + |u|^2) dx d\mathbf{y} \right)^{1/2}.$$

So, by the change of variable theorem,

$$\|u\|_{L^2(Q_\varepsilon)} = \varepsilon^{\frac{d-1}{2}} \|\mathbf{i}_{\boldsymbol{\varepsilon}} u\|_{L^2(Q)},$$

and

$$\|u\|_{H^1(Q_\varepsilon)} = \varepsilon^{\frac{d-1}{2}} \|\mathbf{i}_{\boldsymbol{\varepsilon}} u\|_{H^1_{\boldsymbol{\varepsilon}}(Q)}.$$

Hence, proving this Proposition is equivalent to prove the estimate

$$\|w_\varepsilon - E_\varepsilon v_\varepsilon\|_{H^1(Q_\varepsilon)} \le C\varepsilon \|f_\varepsilon\|_{L^2(Q_\varepsilon)},$$

where w_ε and v_ε are the solutions of the following linear problems, respectivelly,

$$\begin{cases} -\Delta w_\varepsilon + \alpha w_\varepsilon & = f_\varepsilon, \qquad \text{in} \quad Q_\varepsilon \\ \frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} & = 0 \qquad \text{on} \quad \partial Q_\varepsilon, \end{cases} \tag{3.2.3}$$

and

$$\begin{cases} -\frac{1}{g}(g v_{\varepsilon x})_x + \alpha v_\varepsilon & = M_\varepsilon f_\varepsilon, \qquad \text{in} \quad (0,1) \\ v_{\varepsilon x}(0) & = 0, \qquad v_{\varepsilon x}(1) = 0, \end{cases} \tag{3.2.4}$$

with $f_\varepsilon \in L^2(Q_\varepsilon)$. Observe that $u_\varepsilon(x, \mathbf{y}) = w_\varepsilon(x, \varepsilon\mathbf{y})$.

It is known that the minima

$$\lambda_\varepsilon := \min_{\varphi \in H^1(Q_\varepsilon)} \left\{ \frac{1}{2} \int_{Q_\varepsilon} (|\nabla \varphi|^2 + \alpha |\varphi|^2) ds - \int_{Q_\varepsilon} f_\varepsilon \varphi ds \right\}, \tag{3.2.5}$$

$$\tau_\varepsilon := \min_{\varphi \in H^1_g(0,1)} \left\{ \frac{1}{2} \int_0^1 (g|\varphi'|^2 + g\alpha|\varphi|^2)dx - \int_0^1 gM_\varepsilon f_\varepsilon \varphi dx \right\}, \tag{3.2.6}$$

with $s = (x, \mathbf{y}) \in Q_\varepsilon$, are unique and they are attained at the solutions w_ε and v_ε. We want to compare both solutions w_ε and v_ε. We start by taking the function v_ε as a test function in (3.2.5). We have,

$$\lambda_\varepsilon \le \frac{1}{2} \int_{Q_\varepsilon} (|\nabla v_\varepsilon|^2 + \alpha|v_\varepsilon|^2)ds - \int_{Q_\varepsilon} f_\varepsilon v_\varepsilon ds =$$

$$= \frac{1}{2} \int_0^1 \int_{\Gamma_x^\varepsilon} (|v_{\varepsilon x}|^2 + \alpha|v_\varepsilon|^2)d\mathbf{y}dx - \int_0^1 \int_{\Gamma_x^\varepsilon} f_\varepsilon d\mathbf{y} v_\varepsilon dx =$$

$$= \frac{1}{2} \int_0^1 |\Gamma_x^\varepsilon|(|v_{\varepsilon x}|^2 + \alpha|v_\varepsilon|^2)dx - \int_0^1 |\Gamma_x^\varepsilon| M_\varepsilon f_\varepsilon(x, \mathbf{y}) v_\varepsilon dx =$$

$$= \varepsilon^{d-1} \left(\frac{1}{2} \int_0^1 g(x)(|v_{\varepsilon x}|^2 + \alpha|v_\varepsilon|^2)dx - \int_0^1 g(x) M_\varepsilon f_\varepsilon(x, \mathbf{y}) v_\varepsilon dx \right) = \varepsilon^{d-1}\tau_\varepsilon.$$

That is, we have obtained the estimate,

$$\lambda_\varepsilon \le \varepsilon^{d-1}\tau_\varepsilon.$$

To look for a lower bound we proceed as follows,

$$\lambda_\varepsilon = \frac{1}{2} \int_{Q_\varepsilon} (|\nabla w_\varepsilon|^2 + \alpha|w_\varepsilon|^2)ds - \int_{Q_\varepsilon} f_\varepsilon w_\varepsilon ds =$$

$$= \frac{1}{2} \int_{Q_\varepsilon} (|\nabla w_\varepsilon - \nabla v_\varepsilon + \nabla v_\varepsilon|^2 + \alpha|w_\varepsilon - v_\varepsilon + v_\varepsilon|^2)ds - \int_{Q_\varepsilon} f_\varepsilon(w_\varepsilon - v_\varepsilon + v_\varepsilon)ds =$$

$$= \frac{1}{2} \int_{Q_\varepsilon} (|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + |\nabla v_\varepsilon|^2 + 2(\nabla w_\varepsilon - \nabla v_\varepsilon)\nabla v_\varepsilon)ds +$$

$$+ \frac{1}{2} \int_{Q_\varepsilon} \alpha(|w_\varepsilon - v_\varepsilon|^2 + |v_\varepsilon|^2 + 2(w_\varepsilon - v_\varepsilon)v_\varepsilon)ds - \int_{Q_\varepsilon} f_\varepsilon(w_\varepsilon - v_\varepsilon)ds - \int_{Q_\varepsilon} f_\varepsilon v_\varepsilon ds.$$

From above, we know that $\frac{1}{2} \int_{Q_\varepsilon} (|\nabla v_\varepsilon|^2 + \alpha|v_\varepsilon|^2)ds - \int_{Q_\varepsilon} f_\varepsilon v_\varepsilon ds = \varepsilon^{d-1}\tau_\varepsilon$, then

$$\lambda_\varepsilon = \frac{1}{2} \int_{Q_\varepsilon} (|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + 2(\nabla w_\varepsilon - \nabla v_\varepsilon)\nabla v_\varepsilon)ds + \frac{1}{2} \int_{Q_\varepsilon} \alpha(|w_\varepsilon - v_\varepsilon|^2 + 2(w_\varepsilon - v_\varepsilon)v_\varepsilon)ds$$

$$- \int_{Q_\varepsilon} f_\varepsilon(w_\varepsilon - v_\varepsilon)ds + \varepsilon^{d-1}\tau_\varepsilon.$$

To analyze this, we write the last equality like this,

$$\lambda_\varepsilon = \frac{1}{2} \int_{Q_\varepsilon} (|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + \alpha|w_\varepsilon - v_\varepsilon|^2)ds + I_1 + I_2 - I_3 + \varepsilon^{d-1}\tau_\varepsilon,$$

with,

$$I_1 := \int_{Q_\varepsilon} (\nabla w_\varepsilon - \nabla v_\varepsilon)\nabla v_\varepsilon ds, \qquad I_2 := \int_{Q_\varepsilon} \alpha(w_\varepsilon - v_\varepsilon)v_\varepsilon ds,$$

and

$$I_3 := \int_{Q_\varepsilon} f_\varepsilon(w_\varepsilon - v_\varepsilon)ds.$$

If we analyze each term with detail, we observe the following,

$$I_1 = \int_{Q_\varepsilon} (\nabla w_\varepsilon - \nabla v_\varepsilon)\nabla v_\varepsilon ds = \int_{Q_\varepsilon} \left(\frac{\partial w_\varepsilon}{\partial x} - v'_\varepsilon\right) v'_\varepsilon ds =$$

$$\int_{Q_\varepsilon} \left(M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - v'_\varepsilon\right) v'_\varepsilon ds = \int_{Q_\varepsilon} \left(M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)'\right) v'_\varepsilon dx + \int_{Q_\varepsilon} \left((M_\varepsilon w_\varepsilon)' - v'_\varepsilon\right) v'_\varepsilon ds =$$

$$= \int_{Q_\varepsilon} \left(M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)'\right) v'_\varepsilon ds + \varepsilon^{d-1} \int_0^1 g(x)(M_\varepsilon w_\varepsilon - v_\varepsilon)' v'_\varepsilon dx.$$

Since $v_\varepsilon = v_\varepsilon(x)$, we have,

$$I_2 = \int_{Q_\varepsilon} \alpha(w_\varepsilon - v_\varepsilon)v_\varepsilon ds = \varepsilon^{d-1} \int_0^1 \alpha g(x)(M_\varepsilon w_\varepsilon - v_\varepsilon)v_\varepsilon dx,$$

and

$$I_3 = \int_{Q_\varepsilon} (f_\varepsilon - M_\varepsilon f_\varepsilon)(w_\varepsilon - v_\varepsilon)ds + \int_{Q_\varepsilon} M_\varepsilon f_\varepsilon(w_\varepsilon - v_\varepsilon) =$$

$$= \int_{Q_\varepsilon} (f_\varepsilon - M_\varepsilon f_\varepsilon)(w_\varepsilon - v_\varepsilon)ds + \int_{Q_\varepsilon} M_\varepsilon f_\varepsilon(M_\varepsilon w_\varepsilon - v_\varepsilon)ds =$$

$$= \int_{Q_\varepsilon} (f_\varepsilon - M_\varepsilon f_\varepsilon)(w_\varepsilon - v_\varepsilon)ds + \varepsilon^{d-1} \int_0^1 g(x)M_\varepsilon(f_\varepsilon)(M_\varepsilon w_\varepsilon - v_\varepsilon)dx.$$

That is,

$$I_1 = \tilde{I}_1 + \varepsilon^{d-1} \int_0^1 g(x)\left(M_\varepsilon w_\varepsilon - v_\varepsilon\right)' v'_\varepsilon dx, \qquad I_2 = \varepsilon^{d-1} \int_0^1 \alpha g(x)(M_\varepsilon w_\varepsilon - v_\varepsilon)v_\varepsilon dx,$$

and

$$I_3 = \tilde{I}_3 + \varepsilon^{d-1} \int_0^1 g(x)(M_\varepsilon w_\varepsilon - v_\varepsilon)M_\varepsilon f_\varepsilon dx,$$

where

$$\tilde{I}_1 = \int_{Q_\varepsilon} \left(M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)'\right) v'_\varepsilon ds, \qquad \text{and} \qquad \tilde{I}_3 = \int_{Q_\varepsilon} \left(f_\varepsilon - M_\varepsilon f_\varepsilon\right)(w_\varepsilon - v_\varepsilon)ds.$$

We know that,

$$\int_0^1 \left[g(x)(M_\varepsilon w_\varepsilon - v_\varepsilon)' v'_\varepsilon + \alpha g(x)(M_\varepsilon w_\varepsilon - v_\varepsilon)v_\varepsilon\right] dx = \int_0^1 g(x)(M_\varepsilon w_\varepsilon - v_\varepsilon)M_\varepsilon f_\varepsilon dx,$$

then,

$$I_1 + I_2 - I_3 = \tilde{I}_1 - \tilde{I}_3.$$

So, we only need to estimate $\tilde{I}_1$ and $\tilde{I}_3$.

We start with $\tilde{I}_1$.

$$\tilde{I}_1 = \int_{Q_\varepsilon} \left(M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)' \right) v'_\varepsilon ds.$$

Then, we first estimate $M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)'$. For that, we study $(M_\varepsilon w_\varepsilon)'$.

$$(M_\varepsilon w_\varepsilon)' = \frac{d}{dx} \left(\frac{1}{|\Gamma_x^\varepsilon|} \int_{\Gamma_x^\varepsilon} w_\varepsilon(x, \mathbf{y}) d\mathbf{y} \right),$$

and by the Change of Variable Theorem with $\mathbf{y} = \varepsilon \mathbf{L}_x(z)$, see (3.1.1), and $z \in B(0,1)$ the unit ball in $\mathbb{R}^{d-1}$, we have

$$\frac{1}{|\Gamma_x^\varepsilon|} \int_{\Gamma_x^\varepsilon} w_\varepsilon(x, \mathbf{y}) d\mathbf{y} = \int_{B(0,1)} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} dz,$$

where $J_{\mathbf{L}_x(z)}$ is the Jacobian of $\mathbf{L}_x$. So,

$$(M_\varepsilon w_\varepsilon)' = \frac{d}{dx} \left(\int_{B(0,1)} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} dz \right) =$$

$$= \int_{B(0,1)} \frac{\partial w_\varepsilon}{\partial x}(x, \varepsilon \mathbf{L}_x(z)) \frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} dz + \int_{B(0,1)} \nabla_{\mathbf{y}} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \varepsilon \frac{\partial}{\partial x}(\mathbf{L}_x(z)) \frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} dz +$$
$$+ \int_{B(0,1)} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \frac{\partial}{\partial x} \left(\frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} \right) dz.$$

To estimate the right side of the above equality, we study each integral separately. We begin with the first one. Undoing the change of variable $\mathbf{y} = \varepsilon \mathbf{L}_x(z)$,

$$\int_{B(0,1)} \frac{\partial w_\varepsilon}{\partial x}(x, \varepsilon \mathbf{L}_x(z)) \frac{J_{L_x(z)}}{|\Gamma_x^1|} dz = \frac{1}{|\Gamma_x^\varepsilon|} \int_{\Gamma_x^\varepsilon} \frac{\partial w_\varepsilon}{\partial x}(x, \mathbf{y}) d\mathbf{y} = M_\varepsilon \frac{\partial w_\varepsilon}{\partial x}.$$

For the second integral we use $\left| \frac{\partial \mathbf{L}_x(z)}{\partial x} \right| \leq C$,

$$\left| \int_{B(0,1)} \nabla_{\mathbf{y}} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \varepsilon \frac{\partial}{\partial x}(\mathbf{L}_x(z)) \frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} dz \right| \leq C\varepsilon \int_{B(0,1)} |\nabla_{\mathbf{y}} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z))| \frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} dz,$$

undoing again the change of variable, we obtain,

$$\left| \int_{B(0,1)} \nabla_{\mathbf{y}} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \varepsilon \frac{\partial}{\partial x}(\mathbf{L}_x(z)) \frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|} dz \right| \leq C \frac{\varepsilon}{|\Gamma_x^\varepsilon|} \int_{\Gamma_x^\varepsilon} |\nabla_{\mathbf{y}} w_\varepsilon(x, \mathbf{y})| d\mathbf{y}.$$

We estimate the last term as follows,

$$\int_{B(0,1)} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \frac{\partial}{\partial x}\left(\frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|}\right) dz =$$

$$\int_{B(0,1)} \left(w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) - (M_\varepsilon w_\varepsilon)(x)\right) \frac{\partial (J_{\mathbf{L}_x(z)}/|\Gamma_x^1|)}{\partial x}(z) dz +$$

$$+ (M_\varepsilon w_\varepsilon)(x) \int_{B(0,1)} \frac{\partial (J_{\mathbf{L}_x(z)}/|\Gamma_x^1|)}{\partial x}(z) dz.$$

Since,

$$\int_{B(0,1)} \frac{\partial (J_{\mathbf{L}_x(z)}/|\Gamma_x^1|)}{\partial x}(z) dz = \frac{d}{dx}\left(\frac{1}{|\Gamma_x^1|} \underbrace{\int_{B(0,1)} J_{\mathbf{L}_x(z)}(z) dz}_{|\Gamma_x^1|}\right) = 0,$$

then, we have

$$\int_{B(0,1)} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \frac{\partial}{\partial x}\left(\frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|}\right) dz =$$

$$= \int_{B(0,1)} \left(w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) - (M w_\varepsilon)(x)\right) \frac{\partial (J_{\mathbf{L}_x(z)}/|\Gamma_x^1|)}{\partial x}(z) dz.$$

As before, undoing the change of variable and taking account that $\left|\frac{\partial (J_{\mathbf{L}_x(z)}/|\Gamma_x^1|)}{\partial x}\right| \leq C$, we obtain

$$\left|\int_{B(0,1)} w_\varepsilon(x, \varepsilon \mathbf{L}_x(z)) \frac{\partial}{\partial x}\left(\frac{J_{\mathbf{L}_x(z)}}{|\Gamma_x^1|}\right) dz\right| \leq C \frac{1}{|\Gamma_x^\varepsilon|} \int_{\Gamma_x^\varepsilon} |w_\varepsilon(x, \mathbf{y}) - M_\varepsilon w_\varepsilon(x)| d\mathbf{y}.$$

Then, if we put together the three obtained estimates, we have

$$\left| M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)' \right| \leq C \frac{\varepsilon}{|\Gamma_x^\varepsilon|} \int_{\Gamma_x^\varepsilon} \nabla_{\mathbf{y}} w_\varepsilon(x, \mathbf{y}) d\mathbf{y} + C \frac{1}{|\Gamma_x^\varepsilon|} \int_{\Gamma_x^\varepsilon} (w_\varepsilon(x, \mathbf{y}) - M_\varepsilon w_\varepsilon(x)) d\mathbf{y}.$$

So,

$$|\tilde{I}_1| = \left| \int_{Q_\varepsilon} \left(M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - \frac{\partial M_\varepsilon w_\varepsilon}{\partial x} \right) \frac{\partial v_\varepsilon}{\partial x} \right| \leq \int_0^1 C\varepsilon \int_{\Gamma_x^\varepsilon} (\nabla_{\mathbf{y}} w_\varepsilon) v_\varepsilon' d\mathbf{y} dx +$$

$$+ \int_0^1 C \int_{\Gamma_x^\varepsilon} (w_\varepsilon - M_\varepsilon w_\varepsilon) v_\varepsilon' d\mathbf{y} dx.$$

Applying the Hölder inequality, $|\tilde{I}_1|$ can be estimated as follows,

$$\left| \int_{Q_\varepsilon} \left(M_\varepsilon \frac{\partial w_\varepsilon}{\partial x} - \frac{\partial M_\varepsilon w_\varepsilon}{\partial x} \right) \frac{\partial v_\varepsilon}{\partial x} \right| \leq C\varepsilon \|\nabla_{\mathbf{y}} w_\varepsilon\|_{L^2(Q_\varepsilon)} \|v_\varepsilon'\|_{L^2(Q_\varepsilon)} +$$

$$+C\|w_\varepsilon - E_\varepsilon M_\varepsilon w_\varepsilon\|_{L^2(Q_\varepsilon)}\|v'_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

By Lemma 3.1.3,

$$\|w_\varepsilon - E_\varepsilon M_\varepsilon w_\varepsilon\|_{L^2(Q_\varepsilon)} \leq \sqrt{\beta}\varepsilon\|\nabla_{\mathbf{y}} w_\varepsilon\|_{L^2(Q_\varepsilon)},$$

so,

$$\left|\int_{Q_\varepsilon}\left(M_\varepsilon\frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)'\right)v'_\varepsilon ds\right| \leq C\varepsilon\|v'_\varepsilon\|_{L^2(Q_\varepsilon)}\|\nabla_{\mathbf{y}} w_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

To estimate the norm $\|v'_\varepsilon\|_{L^2(Q_\varepsilon)}$ we proceed as follows. We know that v_ε is the solution of

$$\begin{cases} -\frac{1}{g}(gv_{\varepsilon x})_x + \alpha v_\varepsilon &= M_\varepsilon f_\varepsilon, \qquad \text{in} \quad (0,1) \\ \qquad v_{\varepsilon x}(0) &= 0, \qquad v_{\varepsilon x}(1) = 0. \end{cases} \tag{3.2.7}$$

Then, for $x \in (0,1)$, v_ε satisfies,

$$-(gv'_\varepsilon)' + g\alpha v_\varepsilon = gM_\varepsilon f_\varepsilon.$$

If we multiply by v_ε and integrate by parts, we obtain,

$$\int_0^1 g(v'_\varepsilon)^2dx + \alpha\int_0^1 gv_\varepsilon^2dx = \int_0^1(M_\varepsilon f_\varepsilon)gv_\varepsilon dx \leq$$

$$\overset{\text{Hölder ineq.}}{\leq} \left(\int_0^1(M_\varepsilon f_\varepsilon)^2dx\right)^{\frac{1}{2}}\left(\int_0^1(gv_\varepsilon)^2dx\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{4\delta}\int_0^1(M_\varepsilon f_\varepsilon)^2dx + \delta\int_0^1(gv_\varepsilon)^2dx.$$

Then,

$$\int_0^1 g(v'_\varepsilon)^2dx + (\alpha-\delta)\int_0^1 gv_\varepsilon^2dx \leq \frac{1}{4\delta}\int_0^1(M_\varepsilon f_\varepsilon)^2dx \leq \|M_\varepsilon f_\varepsilon\|^2_{L^2(0,1)}$$

So,

$$\int_0^1 g(x)(v'_\varepsilon)^2dx + \int_0^1 gv_\varepsilon^2dx \leq C\|M_\varepsilon f_\varepsilon\|^2_{L^2(0,1)}.$$

And,

$$\|v'_\varepsilon\|^2_{L^2(Q_\varepsilon)} = \int_{Q_\varepsilon}(v'_\varepsilon)^2ds = \int_0^1\int_{\Gamma_x^\varepsilon}(v'_\varepsilon)^2d\mathbf{y}dx = \varepsilon^{d-1}\int_0^1 g(x)(v'_\varepsilon)^2dx \leq$$

$$\leq \varepsilon^{d-1}C\|M_\varepsilon f_\varepsilon\|^2_{L^2(0,1)}.$$

Then,

$$\left|\int_{Q_\varepsilon}\left(M_\varepsilon\frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)'\right)v'_\varepsilon ds\right| \leq C\varepsilon\|v'_\varepsilon\|_{L^2(Q_\varepsilon)}\|\nabla_{\mathbf{y}} w_\varepsilon\|_{L^2(Q_\varepsilon)} \leq$$

$$\leq C\varepsilon\varepsilon^{\frac{d-1}{2}}\|M_\varepsilon f_\varepsilon\|_{L^2(0,1)}\|\nabla_{\mathbf{y}}w_\varepsilon\|_{L^2(Q_\varepsilon)} =$$

$$= C\varepsilon^{\frac{d+1}{2}}\|M_\varepsilon f_\varepsilon\|_{L^2(0,1)}\|\nabla_{\mathbf{y}}w_\varepsilon\|_{L^2(Q_\varepsilon)} \leq C\varepsilon^{d+1}\|M_\varepsilon f_\varepsilon\|^2_{L^2(0,1)} + \frac{1}{4}\|\nabla_{\mathbf{y}}w_\varepsilon\|^2_{L^2(Q_\varepsilon)}.$$

Note that,

$$\|\nabla_{\mathbf{y}}w_\varepsilon\|^2_{L^2(Q_\varepsilon)} \leq \|\nabla w_\varepsilon - \nabla v_\varepsilon\|^2_{L^2(Q_\varepsilon)},$$

so,

$$\left|\int_{Q_\varepsilon}\left(M_\varepsilon\frac{\partial w_\varepsilon}{\partial x} - (M_\varepsilon w_\varepsilon)'\right)v_\varepsilon' ds\right| \leq C\varepsilon^{d+1}\|M_\varepsilon f_\varepsilon\|^2_{L^2(0,1)} + \frac{1}{4}\|\nabla w_\varepsilon - \nabla v_\varepsilon\|^2_{L^2(Q_\varepsilon)}.$$

And $\tilde{I}_3$ can be estimated as follows,

$$\tilde{I}_3 = \int_{Q_\varepsilon}(f_\varepsilon - M_\varepsilon f_\varepsilon)(w_\varepsilon - v_\varepsilon)ds = \int_{Q_\varepsilon}(f_\varepsilon - M_\varepsilon f_\varepsilon)(w_\varepsilon - M_\varepsilon w_\varepsilon)ds +$$

$$+ \underbrace{\int_{Q_\varepsilon}(f_\varepsilon - M_\varepsilon f_\varepsilon)(M_\varepsilon w_\varepsilon - v_\varepsilon)ds}_{=0},$$

by the Hölder inequality,

$$|\tilde{I}_3| = \left|\int_{Q_\varepsilon}(f_\varepsilon - M_\varepsilon f_\varepsilon)(w_\varepsilon - M_\varepsilon w_\varepsilon)ds\right| \leq \|f_\varepsilon - M_\varepsilon f_\varepsilon\|_{L^2(Q_\varepsilon)}\|w_\varepsilon - M_\varepsilon w_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

Again, by Lemma 3.1.3,

$$\|w_\varepsilon - M_\varepsilon w_\varepsilon\|^2_{L^2(Q_\varepsilon)} \leq \beta\varepsilon^2\|\nabla_{\mathbf{y}}w_\varepsilon\|^2_{L^2(Q_\varepsilon)},$$

so,

$$|\tilde{I}_3| = \left|\int_{Q_\varepsilon}(f_\varepsilon - M_\varepsilon f_\varepsilon)(w_\varepsilon - M_\varepsilon w_\varepsilon)ds\right| \leq \|f_\varepsilon - M_\varepsilon f_\varepsilon\|_{L^2(Q_\varepsilon)}\sqrt{\beta}\varepsilon\|\nabla_{\mathbf{y}}w_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

If we join all the estimates, then

$$\lambda_\varepsilon = \frac{1}{2}\int_{Q_\varepsilon}|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + |w_\varepsilon - v_\varepsilon|^2 + \varepsilon^{d-1}\tau_\varepsilon + \theta_\varepsilon,$$

where,

$$|\theta_\varepsilon| = |\tilde{I}_1 - \tilde{I}_3| \leq C\varepsilon^{d+1}\|M_\varepsilon f_\varepsilon\|^2_{L^2(0,1)} + \frac{1}{4}\|\nabla w_\varepsilon - \nabla v_\varepsilon\|^2_{L^2(Q_\varepsilon)} +$$

$$+ \|f_\varepsilon - M_\varepsilon f_\varepsilon\|_{L^2(Q_\varepsilon)}\sqrt{\beta}\varepsilon\|\nabla_{\mathbf{y}}w_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

With this,

$$\lambda_\varepsilon \geq \frac{1}{2}\int_{Q_\varepsilon}(|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + |w_\varepsilon - v_\varepsilon|^2)ds + \varepsilon^{d-1}\tau_\varepsilon - C\varepsilon^{d+1}\|M_\varepsilon f_\varepsilon\|^2_{L^2(0,1)}$$

$$-\frac{1}{4}\|\nabla w_\varepsilon - \nabla v_\varepsilon\|^2_{L^2(Q_\varepsilon)} - \|f_\varepsilon - M_\varepsilon f_\varepsilon\|_{L^2(Q_\varepsilon)}\sqrt{\beta}\varepsilon\|\nabla_{\mathbf{y}} w_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

By Lemma 3.1.3 $\|M_\varepsilon f_\varepsilon\|_{L^2_g(0,1)} \leq \varepsilon^{\frac{1-d}{2}}\|f_\varepsilon\|_{L^2(Q_\varepsilon)}$, then

$$\lambda_\varepsilon \geq \frac{1}{4}\int_{Q_\varepsilon}(|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + |w_\varepsilon - v_\varepsilon|^2)ds +$$

$$+\varepsilon^{d-1}\tau_\varepsilon - C\varepsilon^2\|f_\varepsilon\|^2_{L^2(Q_\varepsilon)} - \|f_\varepsilon - M_\varepsilon f_\varepsilon\|_{L^2(Q_\varepsilon)}\sqrt{\beta}\varepsilon\|\nabla_{\mathbf{y}} w_\varepsilon\|_{L^2(Q_\varepsilon)}.$$

If we put everything together,

$$\varepsilon^{d-1}\tau_\varepsilon \geq \lambda_\varepsilon \geq \frac{1}{4}\int_{Q_\varepsilon}(|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + |w_\varepsilon - v_\varepsilon|^2)ds + \varepsilon^{d-1}\tau_\varepsilon - C\varepsilon^2\|f_\varepsilon\|^2_{L^2(Q_\varepsilon)} -$$

$$-\|f_\varepsilon - M_\varepsilon f_\varepsilon\|_{L^2(Q_\varepsilon)}\sqrt{\beta}\varepsilon\|\nabla_{\mathbf{y}} w_\varepsilon\|_{L^2(Q_\varepsilon)} \geq$$

$$\geq \frac{1}{4}\int_{Q_\varepsilon}(|\nabla w_\varepsilon - \nabla v_\varepsilon|^2 + |w_\varepsilon - v_\varepsilon|^2)ds + \varepsilon^{d-1}\tau_\varepsilon - C\varepsilon^2\|f_\varepsilon\|^2_{L^2(Q_\varepsilon)} -$$

$$\frac{\beta\varepsilon^2}{2}\|f_\varepsilon - M_\varepsilon f_\varepsilon\|^2_{L^2(Q_\varepsilon)} - \frac{1}{2}\|\nabla_{\mathbf{y}} w_\varepsilon\|^2_{L^2(Q_\varepsilon)},$$

so,

$$\|\nabla w_\varepsilon - \nabla v_\varepsilon\|^2_{L^2(Q_\varepsilon)} + \|w_\varepsilon - v_\varepsilon\|^2_{L^2(Q_\varepsilon)} \leq C\varepsilon^2\|f_\varepsilon\|^2_{L^2(Q_\varepsilon)} + \frac{\beta\varepsilon^2}{2}\|f_\varepsilon - M_\varepsilon f_\varepsilon\|^2_{L^2(Q_\varepsilon)},$$

and so,

$$\|w_\varepsilon - E_\varepsilon v_\varepsilon\|_{H^1(Q_\varepsilon)} \leq C\varepsilon\|f_\varepsilon\|_{L^2(Q_\varepsilon)},$$

that is,

$$\|u_\varepsilon - Ev_\varepsilon\|_{H^1_\varepsilon(Q)} \leq C\varepsilon\|f_\varepsilon\|_{L^2(Q)}. \tag{3.2.8}$$

■

Remark 3.2.2. *Note that if we consider problems* (3.2.1) *and* (3.2.2) *with* $f_\varepsilon = Eh_\varepsilon$ *then,* $\|f_\varepsilon - M_\varepsilon f_\varepsilon\|_{L^2(Q_\varepsilon)} = 0$ *and so, we obtain the same estimate,*

$$\|u_\varepsilon - Ev_\varepsilon\|_{H^1_\varepsilon(Q)} \leq C\varepsilon\|Eh_\varepsilon\|_{L^2(Q)}.$$

We show now, in a formal way, that this estimate obtained in Proposition 3.2.1 is optimal. For this, we will consider a domain Q_ε having circular cross sections and with the aid of an asymptotic expansion of the solution u_ε, we will obtain that the estimates obtained are optimal.

Hence, let $Q_\varepsilon = \{(x, \varepsilon\mathbf{y}) \in \mathbb{R}^d : (x, \mathbf{y}) \in Q\}$, with $\varepsilon \in (0, 1)$, and $Q = \{(x, \mathbf{y}) \in \mathbb{R}^d : 0 \leq x \leq 1, |\mathbf{y}| < r(x)\}$, so that the transversal sections Γ^1_x of the domain Q are disks centered at the origin of radius $r(x)$. Obviously, the change of variables which takes Q_ε into the fixed domain Q is the following,

$$X = x, \qquad \mathbf{Y} = \varepsilon \mathbf{y},$$

with $(X, \mathbf{Y}) \in Q_\varepsilon$ and $(x, \mathbf{y}) \in Q$. This change of variables transforms the original problem into the following linear problem in Q (we consider the coefficient of equation $\alpha = 1$),

$$\begin{cases} -\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{1}{\varepsilon^2}\Delta_{\mathbf{y}} u_\varepsilon + u_\varepsilon & = Ef, \qquad \text{in} \quad Q \\ (\nabla_x u_\varepsilon, \frac{1}{\varepsilon}\nabla_{\mathbf{y}} u_\varepsilon) \cdot \nu_\varepsilon & = 0 \qquad \text{on} \quad \partial Q, \end{cases} \tag{3.2.9}$$

with $\partial Q = \Gamma_0^1 \cup \partial_l Q \cup \Gamma_1^1$, where $\partial_l Q$ is the "lateral boundary" which is given by $\partial_l Q = \{(x, \mathbf{y}) : \mathbf{y} \in \partial \Gamma_x^1\}$ and

$$\nu_\varepsilon = \begin{cases} (-1, 0), & \text{in} \quad \Gamma_0^1, \\ \left(\frac{-\varepsilon r r'}{r\sqrt{\varepsilon^2 r'^2 + 1}}, \frac{\mathbf{y}}{r\sqrt{\varepsilon^2 r'^2+1}} \right), & \text{in} \quad \partial\Gamma_x^1, \\ (1, 0), & \text{in} \quad \Gamma_1^1. \end{cases} \tag{3.2.10}$$

The limit problem is given by

$$\begin{cases} -\frac{1}{g}(g v_{0x})_x + v_0 & = f, \qquad \text{in} \quad (0,1) \\ v_{0x}(0) & = 0, \qquad v_{0x}(1) = 0, \end{cases} \tag{3.2.11}$$

with $f \in L_g^2(0,1)$. Recall that $g(x) = |\Gamma_x^1| = r(x)^{d-1}\omega_{d-1}$ and ω_{d-1} is the $(d-1)$-measure of the unit ball in $\mathbb{R}^{d-1}$.

To analyze the rate of convergence of $u_\varepsilon \to v_0$ as $\varepsilon \to 0$, we express the solution of (3.2.9) as the series

$$u_\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i V_i(x, \mathbf{y}) = V_0(x, \mathbf{y}) + \varepsilon V_1(x, \mathbf{y}) + \varepsilon^2 V_2(x, \mathbf{y}) + \dots$$

Introducing this expression in problem (3.2.9) we obtain,

$$\begin{cases} -\sum_{i=0}^{\infty} \varepsilon^i V_{i_{xx}} - \frac{1}{\varepsilon^2}\sum_{i=0}^{\infty} \varepsilon^i \Delta_{\mathbf{y}} V_i + \sum_{i=0}^{\infty} \varepsilon^i V_i(x, \mathbf{y}) & = Ef, \qquad \text{in} \quad Q \\ -\sum_{i=0}^{\infty} \varepsilon^i V_{i_x} & = 0, \qquad \text{on} \quad \Gamma_0^1 \\ -\sum_{i=0}^{\infty} \varepsilon^{i+1} V_{i_x} r r' + \sum_{i=0}^{\infty} \varepsilon^{i-1} \nabla_{\mathbf{y}} V_i \cdot \mathbf{y} & = 0 \qquad \text{on} \quad \partial\Gamma_x^1 \\ \sum_{i=0}^{\infty} \varepsilon^i V_{i_x} & = 0, \qquad \text{on} \quad \Gamma_1^1 \end{cases} \tag{3.2.12}$$

Putting in groups of powers of ε, we have the following equalities in Q,

$$
\begin{aligned}
&\Delta_{\mathbf{y}} V_0(x,\mathbf{y}) = 0,\\
&\Delta_{\mathbf{y}} V_1(x,\mathbf{y}) = 0,\\
&-V_{0_{xx}}(x,\mathbf{y}) - \Delta_{\mathbf{y}} V_2(x,\mathbf{y}) + V_0(x,\mathbf{y}) - f(x) = 0,\\
&-V_{i_{xx}}(x,\mathbf{y}) - \Delta_{\mathbf{y}} V_{i+2}(x,\mathbf{y}) + V_i(x,\mathbf{y}) = 0, \qquad \text{for} \quad i = 1,2,...
\end{aligned}
\tag{3.2.13}
$$

and, from the boundary condition, we have,

$$
\begin{aligned}
&V_{i_x}(x,\mathbf{y}) = 0 \qquad \text{on} \quad \Gamma_0^1 \cup \Gamma_1^1, \qquad \text{for} \quad i = 0,1,2,...\\
&\nabla_{\mathbf{y}} V_0(x,\mathbf{y}) \cdot \mathbf{y} = 0, \qquad \text{on } \partial\Gamma_x^1\\
&\nabla_{\mathbf{y}} V_1(x,\mathbf{y}) \cdot \mathbf{y} = 0, \qquad \text{on } \partial\Gamma_x^1\\
&-V_{i_x}(x,\mathbf{y}) r r' + \nabla_{\mathbf{y}} V_{i+2}(x,\mathbf{y}) \cdot \mathbf{y} = 0, \qquad \text{for} \quad i = 0,1,2,... \quad \text{on } \partial\Gamma_x^1
\end{aligned}
\tag{3.2.14}
$$

First, for $x \in (0,1)$ fixed, we focus in the particular problems in Γ_x^1 in which $V_0(x,\mathbf{y})$ and $V_1(x,\mathbf{y})$ are involved,

$$
\begin{array}{lllllll}
\Delta_{\mathbf{y}} V_0(x,\mathbf{y}) & = 0 & \text{in} \quad \Gamma_x^1, & \qquad & \Delta_{\mathbf{y}} V_1(x,\mathbf{y}) & = 0 & \text{in} \quad \Gamma_x^1\\
\nabla_{\mathbf{y}} V_0(x,\mathbf{y}) \cdot \mathbf{y} & = 0 & \text{on} \quad \partial\Gamma_x^1, & & \nabla_{\mathbf{y}} V_1(x,\mathbf{y}) \cdot \mathbf{y} & = 0 & \text{on} \quad \partial\Gamma_x^1.
\end{array}
\tag{3.2.15}
$$

Both problems imply that, for each $x \in (0,1)$, $V_0(x,\mathbf{y})$ and $V_1(x,\mathbf{y})$ are constant in Γ_x^1. It means both functions only depend on x,

$$
V_0(x,\mathbf{y}) = V_0(x), \qquad\qquad V_1(x,\mathbf{y}) = V_1(x).
$$

Since V_0 only depends on x, the third condition in (3.2.13) and in (3.2.14) can be written as

$$
\left\{
\begin{array}{llr}
\Delta_{\mathbf{y}} V_2(x,\mathbf{y}) & = -V_{0_{xx}}(x) + V_0(x) - f(x) & \text{in} \quad \Gamma_x^1,\\
\nabla_{\mathbf{y}} V_2(x,\mathbf{y}) \cdot \nu & = V_{0_x}(x) r'(x) & \text{on} \quad \partial\Gamma_x^1.
\end{array}
\right.
\tag{3.2.16}
$$

Integrating over Γ_1^x in the equation and using the boundary condition, we find that in order to have solutions of (3.2.16) we must have (Fredholm alternative),

$$
V_{0x} r' |\partial\Gamma_x^1| = (-V_{0_{xx}}(x) + V_0(x) - f(x)) |\Gamma_x^1|.
$$

That is,

$$
-V_{0_{xx}}(x) + V_0(x) - f(x) = V_{0x} r' \frac{|\partial\Gamma_x^1|}{|\Gamma_x^1|} = \frac{d-1}{r} V_{0_x} r'.
$$

Now, since $\frac{g_x}{g} = (d-1)\frac{r'}{r}$ we easily get

$$
-\frac{1}{g}(g V_{0_x})_x + V_0 = f(x),
$$

and the boundary conditions are given by

$$V_{0_x}(0) = V_{0_x}(1) = 0.$$

This implies $V_0(x, \mathbf{y}) = v_0(x)$ is the solution of the limit problem (3.2.11). Moreover, the function $V_2(x, \mathbf{y})$ satisfies (3.2.16) and it is not identically 0 in general (if for instance $f \not\equiv 0$).

Proceeding in a similar way with V_1 and V_3 we get,

$$\begin{cases} \Delta_{\mathbf{y}} V_3(x, \mathbf{y}) &= -V_{1_{xx}}(x) + V_1(x) & \text{in} \quad \Gamma_x^1, \\ \nabla_{\mathbf{y}} V_3(x, \mathbf{y}) \cdot \nu &= r' V_{1_x}(x) & \text{on} \quad \partial\Gamma_x^1, \end{cases} \tag{3.2.17}$$

and with the Fredholm alternative, the function V_1 needs to satisfy $-\frac{1}{g}(gV_{1_x})_x + V_1 = 0$, with the boundary conditions $V_{1x}(0) = V_{1x}(1) = 0$ (see (3.2.14)). This implies that $V_1(\cdot) \equiv 0$ and from (3.2.17) we get $V_3 = V_3(x)$. With an induction argument it is not difficult to see now that $V_i \equiv 0$ for all odd i. Hence,

$$u_\varepsilon(x, \mathbf{y}) = v_0(x) + \varepsilon^2 V_2(x, \mathbf{y}) + \varepsilon^4 V_4(x, \mathbf{y}) + \ldots$$

where $V_2(x, \mathbf{y})$ is the solution of (3.2.16) which is generically non zero.

Then, for ε small enough,

$$\|u_\varepsilon - EV_0\|_{H^1_\varepsilon(Q)} = \|\varepsilon^2 V_2(x, \mathbf{y})\|_{H^1_\varepsilon(Q)} + \ldots$$

$$= \left(\varepsilon^4 \int_Q (|\nabla_x V_2|^2 + \frac{1}{\varepsilon^2}|\nabla_{\mathbf{y}} V_2|^2 + |V_2|^2) dx d\mathbf{y} \right)^{\frac{1}{2}} + \ldots = \varepsilon \|\nabla_{\mathbf{y}} V_2\|_{L^2(Q)} + o(\varepsilon).$$

But,

$$\|\nabla_{\mathbf{y}} V_2\|_{L^2(Q)} \sim \|f\|_{L^2(Q)},$$

which implies that estimate from Proposition 3.2.1 is optimal.

3.3. Analysis of the nonlinear terms

In this section we focus our study in the nonlinear terms. We will analyze its differentiability properties and we will prepare the nonlinearities to apply the results on inertial manifolds from the previous chapter. As a matter of fact, we will show that with appropriate cut-off functions the new nonlinearities satisfy hypotheses **(H2)** and **(H2')** from Chapter 2, easing our way to the construction of the inertial manifolds and to estimating the distance between them.

Throughout this section, we consider X^α_ε, for $0 \leq \varepsilon \leq \varepsilon_0$ and $0 < \alpha < \frac{1}{2}$, the fractional power space, see [19] and [32], corresponding to elliptic problem (3.2.1) and (3.2.2).

First, we analyze the properties the nonlinear terms satisfy. Remember that the nonlinearity f, together with its first and second derivative satisfy the boundedness condition (3.1.10). We denote by $F_\varepsilon : X^\alpha_\varepsilon \to L^2(Q)$ the Nemytskii operator corresponding to f, that is,

$$\begin{aligned} F_\varepsilon : X^\alpha_\varepsilon &\longrightarrow L^2(Q), \\ u &\longmapsto f(u), \end{aligned} \tag{3.3.1}$$

$$\begin{aligned} F_0 : X^\alpha_0 &\longrightarrow L^2_g(0,1), \\ u &\longmapsto f(u). \end{aligned} \tag{3.3.2}$$

Then we have the following result.

Lemma 3.3.1. *The Nemytskii operator F_ε, $\varepsilon \geq 0$, satisfies the following properties:*

(i) F_ε is uniformly bounded from X^α_ε into $L^2(Q)$. That is, there exists a constant $C_F > 0$ independent of ε such that,

$$\|F_\varepsilon\|_{L^\infty(X^\alpha_\varepsilon, L^2(Q))} \leq C_F.$$

(ii) There exists $\theta_F \in (0,1]$ such that F_ε is $C^{1,\theta_F}(X^\alpha_\varepsilon, L^2(Q))$ uniformly in ε. That is, there exists a constant $L_F > 0$, such that,

$$\|F_\varepsilon(u) - F_\varepsilon(v)\|_{L^2(Q)} \leq L_F \|u - v\|_{X^\alpha_\varepsilon},$$

$$\|DF_\varepsilon(u) - DF_\varepsilon(v)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} \leq L_F \|u - v\|^{\theta_F}_{X^\alpha_\varepsilon}$$

for all $u, v \in X^\alpha_\varepsilon$ and all $0 \leq \varepsilon \leq \varepsilon_0$:

Proof. Item (i) is directly proved as follows. Since nonlinearity f is uniformly bounded, see (3.1.10),

$$\|F_\varepsilon\|_{L^\infty(X^\alpha_\varepsilon, L^2(Q))} = \sup_{u \in X^\alpha_\varepsilon} \left(\int_Q |f(u(x, \mathbf{y}))|^2 dx d\mathbf{y} \right)^{\frac{1}{2}} \leq L_f |Q|^{\frac{1}{2}},$$

for any $\varepsilon \geq 0$ and $|Q|$ the Lebesgue measure of Q. So, we have the desired estimate with $C_F = L_f |Q|^{\frac{1}{2}}$.

To prove item (ii), we proceed as follows.

$$\|F_\varepsilon(u) - F_\varepsilon(v)\|_{L^2(Q)} = \left(\int_Q |f(u(x, \mathbf{y})) - f(v(x, \mathbf{y}))|^2 dx d\mathbf{y} \right)^{\frac{1}{2}}.$$

Since f is globally Lipschitz, see (3.1.10), then,

$$\|F_\varepsilon(u) - F_\varepsilon(v)\|_{L^2(Q)} \le L_f \left(\int_Q |u(x,\mathbf{y}) - v(x,\mathbf{y})|^2 dx d\mathbf{y} \right)^{\frac{1}{2}} =$$

$$= L_f \|u - v\|_{L^2(Q)} \le L_f \|u - v\|_{X^\alpha_\varepsilon},$$

taking $L_F = L_f$ we have, for all $\varepsilon \ge 0$, that F_ε is globally Lipschitz from X^α_ε into $L^2(Q)$ with uniform constant L_F. To show the remaining part, notice first that for $u \in X^\alpha_\varepsilon$, $DF_\varepsilon(u)$ is given by the operator

$$\begin{aligned} DF_\varepsilon(u) : X^\alpha_\varepsilon &\longrightarrow L^2(Q), \\ v &\longmapsto f'(u)v, \end{aligned} \tag{3.3.3}$$

which is easily shown from the definition of Fréchet derivative, the Sobolev embeddings $X^\alpha_\varepsilon \hookrightarrow L^q$ for $q > 2$, and the property (3.1.10). That is,

$$\|F_\varepsilon(u+v) - F_\varepsilon(u) - f'(u)v\|_{L^2(Q)} = \| \left(f'(\xi) - f'(u) \right) v \|_{L^2(Q)},$$

with ξ an intermediate point between u and $u + v$.

But, by (3.1.10) $|f'(\xi) - f'(u)| \le 2L_f$ and also by the mean value theorem $|f'(\xi) - f'(u)| \le L_f |\xi - u| \le L_f |v|$. This implies $|f'(\xi) - f'(u)| \le 2L_f |v|^\theta$, for all $0 < \theta < 1$.

Hence,

$$\|F_\varepsilon(u+v) - F_\varepsilon(u) - f'(u)v\|_{L^2(Q)} \le 2L_f \|v^{1+\theta}\|_{L^2(Q)} = 2L_f \|v\|^{1+\theta}_{L^{2+2\theta}(Q)}.$$

Choosing $2 + 2\theta < q$ we get that $DF_\varepsilon(u)v = f'(u)v$.

Moreover, we have that, for all $\varepsilon \ge 0$,

$$\|DF_\varepsilon(u) - DF_\varepsilon(v)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} = \sup_{\phi \in X^\alpha_\varepsilon,\ \|\phi\|_{X^\alpha_\varepsilon} \le 1} \|DF_\varepsilon(u)\phi - DF_\varepsilon(v)\phi\|_{L^2(Q)}.$$

Hence,

$$\|DF_\varepsilon(u) - DF_\varepsilon(v)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} = \sup_{\phi \in X^\alpha_\varepsilon,\ \|\phi\|_{X^\alpha_\varepsilon} \le 1} \left(\int_Q (f'(u) - f'(v))^2 \phi^2 dx d\mathbf{y} \right)^{\frac{1}{2}}.$$

Note that, by Hölder inequality with exponents $\frac{d}{4\alpha}$ and $\frac{d}{d-4\alpha}$, (remember $\alpha < \frac{1}{2}$ and $d \ge 2$, so that both $\frac{d}{4\alpha}$, $\frac{d}{d-4\alpha} \in (1, \infty)$), we have,

$$\int_Q (f'(u) - f'(v))^2 \phi^2 dx d\mathbf{y} \le \left(\int_Q |f'(u) - f'(v)|^{\frac{d}{2\alpha}} dx d\mathbf{y} \right)^{\frac{4\alpha}{d}} \left(\int_Q |\phi|^{\frac{2d}{d-4\alpha}} dx d\mathbf{y} \right)^{\frac{d-4\alpha}{d}}.$$

Observe that X^α_ε is the fractional power space of a selfadjoint operator in a Hilbert setting. Hence it is possible to show that we have the embedding $X^\alpha_\varepsilon \hookrightarrow L^{\frac{2d}{d-4\alpha}}$ and that we have $\|\phi\|_{L^{\frac{2d}{d-4\alpha}}} \leq C_\varepsilon \|\phi\|_{X^\alpha_\varepsilon}$. Moreover a deeper analysis shows that this constant C_ε can be chosen uniformly in ε. This is a non trivial fact at all since we are dealing with a class of selfadjoint operators which are singularly perturbed (a factor $\frac{1}{\varepsilon}$ appears in the coefficient of the operators). We have included a proof of this fact, which deals with interpolation theory, in Section 3.6.

Then we have,

$$\int_Q (f'(u) - f'(v))^2 \phi^2 dxd\mathbf{y} \leq C \left(\int_Q |f'(u) - f'(v)|^{\frac{d}{2\alpha}} dxd\mathbf{y} \right)^{\frac{4\alpha}{d}} \|\phi\|^2_{X^\alpha_\varepsilon}.$$

Then,

$$\sup_{\phi \in X^\alpha_\varepsilon,\ \|\phi\|_{X^\alpha_\varepsilon} \leq 1} \left(\int_Q (f'(u) - f'(v))^2 \phi^2 dxd\mathbf{y} \right)^{\frac{1}{2}} \leq \left(\int_Q |f'(u) - f'(v)|^{\frac{d}{2\alpha}} dxd\mathbf{y} \right)^{\frac{2\alpha}{d}}$$

Next, note that, on the one side, by the mean value theorem and using 3.1.10, we have,

$$|f'(u) - f'(v)| \leq L_f |u - v|.$$

On the other side, again by (3.1.10),

$$|f'(u) - f'(v)| \leq 2L_f.$$

Hence,

$$|f'(u) - f'(v)| \leq 2L_f \min\{1, |u - v|\} \leq 2L_f |u - v|^\theta,$$

for any $0 \leq \theta \leq 1$, where we have used that if $0 \leq x \leq 1$ and $0 \leq \theta \leq 1$ then $x \leq x^\theta$. Then,

$$\|DF_\varepsilon(u) - DF_\varepsilon(v)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} \leq 2L_f \left(\int_Q |u - v|^{\frac{\theta d}{2\alpha}} dxd\mathbf{y} \right)^{\frac{2\alpha}{d}}.$$

Taking $\theta_F = \min\{1, \frac{4\alpha}{d-4\alpha}\}$,

$$\|DF_\varepsilon(u) - DF_\varepsilon(v)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} \leq 2L_f \left(\int_Q |u - v|^{\frac{2d}{d-4\alpha}} dxd\mathbf{y} \right)^{\frac{2\alpha}{d}} = 2L_f \|u - v\|^{\theta_F}_{L^{\frac{2d}{d-4\alpha}}(Q)}.$$

Applying again the uniform embedding described in Section 3.6, we obtain

$$\|DF_\varepsilon(u) - DF_\varepsilon(v)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} \leq 2L_f \|u - v\|^{\theta_F}_{X^\alpha_\varepsilon}.$$

Taking $L_F = 2L_f$ we have the result.

■

Remark 3.3.2. *i) Note that we have to impose α strictly positive to guarantee the smoothness of F_ε, that is, to ensure that $F_\varepsilon \in C^{1,\theta}(X^\alpha_\varepsilon, L^2(Q))$ for θ small enough. As a matter of fact if $\alpha = 0$ $X^\alpha_\varepsilon = L^2(Q)$, any nonlinearity $F : L^2(Q) \to L^2(Q)$ which is a Nemytskii operator, as in (3.3.1), cannot be C^1, unless it is linear, see [32], Exercise 1. Although in [32], the author considers the case $F_\varepsilon(u) = sin(u)$, the argument can be easily extended to any C^2 function.*

ii) if $d \geq 4$ we always have that $\theta_F = \frac{4\alpha}{d-4\alpha} < 1$ because $\alpha < 1/2$. Only in dimensions $d = 2, 3$ and choosing $\alpha < 1/2$ but close enough to $1/2$ we may get $\frac{4\alpha}{d-4\alpha} > 1$ and therefore $\theta_F = 1$. As a matter of fact, in dimensions $d = 2, 3$ we may show some higher differentiability of F.

We fix α with $0 < \alpha < \frac{1}{2}$.

As we have mentioned above, one of our basic tools consists in constructing inertial manifolds to reduce our problem to a finite dimensional one. In order to construct these manifolds and following [52], we need to "prepare" the non-linear term making an appropriate cut off of the nonlinearity in the X^α_ε norm, as it is done in [52] .

Next, we proceed to introduce this cut off. For this, we start considering a function $\hat{\Theta} : \mathbb{R} \to [0, 1]$ which is C^∞ with compact support and such that

$$\hat{\Theta}(x) = \begin{cases} 1 & if \quad |x| \leq R^2 \\ 0 & if \quad |x| \geq 4R^2. \end{cases} \tag{3.3.4}$$

for some $R > 0$, which in general will be large enough. We will denote this function $\hat{\Theta}^R(x)$ if we need to make explicit its dependence on the parameter R. With this function we define now $\Theta_\varepsilon : X^\alpha_\varepsilon \to \mathbb{R}$ as $\Theta_\varepsilon(u) = \hat{\Theta}(\|u\|^2_{X^\alpha_\varepsilon})$ for $0 \leq \varepsilon \leq \varepsilon_0$, and observe that $\Theta_\varepsilon(u) = 1$ if $\|u\|_{X^\alpha_\varepsilon} \leq R$ and $\Theta_\varepsilon(u) = 0$ if $\|u\|_{X^\alpha_\varepsilon} \geq 2R$ and again we will denote Θ_ε by Θ^R_ε if we need to make explicit its dependence on R.

Now, for $R > 0$, large enough, and $0 < \varepsilon \leq \varepsilon_0$, we introduce the new nonlinear terms

$$\tilde{F}_\varepsilon(u_\varepsilon) := \Theta^R_\varepsilon(u_\varepsilon)F_\varepsilon(u_\varepsilon), \tag{3.3.5}$$

$$\tilde{F}^\varepsilon_0(u_0) := \Theta^R_\varepsilon(Eu_0)F_0(u_0), \tag{3.3.6}$$

and

$$\tilde{F}_0(u_0) := \Theta^R_0(u_0)F_0(u_0), \tag{3.3.7}$$

We replace F_ε and F_0 with the new nonlinearities $\tilde{F}_\varepsilon$, $\tilde{F}^\varepsilon_0$ and $\tilde{F}_0$. Hence, now we have three systems, two of them in the limit space X^α_0,

$$u_t = -A_\varepsilon u + \tilde{F}_\varepsilon(u), \qquad u \in X^\alpha_\varepsilon \tag{3.3.8}$$

$$u_t = -A_0 u + \tilde{F}^\varepsilon_0(u), \qquad u \in X^\alpha_0, \tag{3.3.9}$$

$$u_t = -A_0 u + \tilde{F}_0(u), \qquad u \in X_0^\alpha. \tag{3.3.10}$$

Note that, since systems (3.3.9) and (3.3.10) share the linear part and $\tilde{F}_0(u) = \tilde{F}_0^\varepsilon(u)$ for $\|u\|_{X_0^\alpha} \leq R$, then the attractor related to (3.3.9) and (3.3.10) coincides and it is $\mathcal{A}_0$. Moreover, although $\tilde{F}_0^\varepsilon, \tilde{F}_0 : X_0^\alpha \to X_0$, the nonlinearity $\tilde{F}_0^\varepsilon$ depends on ε.

Remark 3.3.3. *It may sound somehow strange the need to consider now three systems instead of the natural two (the perturbed one* (3.3.8) *and the completely unperturbed one* (3.3.10)*). The three systems meet the conditions to have inertial manifolds and we will see that they all are nearby in the C^1 topology. But, as we will see below, we will have good estimates for the distance between the inertial manifolds for systems* (3.3.8) *and* (3.3.9) *but not so good estimates for the distance between the inertial manifolds for systems* (3.3.8) *and* (3.3.10) *or* (3.3.9) *and* (3.3.10)*.*

First, we analyze the properties $\tilde{F}_\varepsilon$, $\tilde{F}_0^\varepsilon$ and $\tilde{F}_0$ satisfy.

Lemma 3.3.4. *Let $\tilde{F}_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, $\tilde{F}_0^\varepsilon$ and $\tilde{F}_0$, be the new nonlinearities described above. Then they satisfy the following properties:*

(a) $\tilde{F}_\varepsilon(u) = F_\varepsilon(u)$, for all $u \in X_\varepsilon^\alpha$, such that $\|u\|_{X_\varepsilon^\alpha} \leq R$, $\varepsilon > 0$ and $\tilde{F}_0^\varepsilon(u_0) = F_0(u_0)$, $\tilde{F}_0(u_0) = F_0(u_0)$, for all $u_0 \in X_0^\alpha$, such that $\|Eu_0\|_{X_\varepsilon^\alpha} \leq R$ and $\|u_0\|_{X_0^\alpha} \leq R$, respectively.

(b) $\tilde{F}_\varepsilon$ is $C^{1,\theta_F}(X_\varepsilon^\alpha, L^2(Q))$ and $\tilde{F}_0^\varepsilon$, $\tilde{F}_0$ are $C^{1,\theta_F}(X_0^\alpha, L_g^2(0,1))$ with θ_F the one from Lemma 3.3.1. That is, they are globally Lipschitz from X_ε^α to $L^2(Q)$ and from X_0^α to $L_g^2(0,1)$, we denote by L_F their Lipschitz constant, and

$$\|D\tilde{F}_\varepsilon(u) - D\tilde{F}_\varepsilon(u')\|_{\mathcal{L}(X_\varepsilon^\alpha, L^2(Q))} \leq L_F \|u - u'\|_{X_\varepsilon^\alpha}^{\theta_F}, \tag{3.3.11}$$

$$\|D\tilde{F}_0^\varepsilon(u) - D\tilde{F}_0^\varepsilon(u')\|_{\mathcal{L}(X_0^\alpha, L_g^2(0,1))} \leq L_F \|u - u'\|_{X_0^\alpha}^{\theta_F}, \tag{3.3.12}$$

$$\|D\tilde{F}_0(u) - D\tilde{F}_0(u')\|_{\mathcal{L}(X_0^\alpha, L_g^2(0,1))} \leq L_F \|u - u'\|_{X_0^\alpha}^{\theta_F}, \tag{3.3.13}$$

with L_F independent of ε.

(c) They are uniformly bounded,

$$\|\tilde{F}_\varepsilon\|_{L^\infty(X_\varepsilon^\alpha, L^2(Q))} \leq C_F, \quad \|\tilde{F}_0^\varepsilon\|_{L^\infty(X_0^\alpha, L_g^2(0,1))} \leq C_F, \quad \|\tilde{F}_0\|_{L^\infty(X_0^\alpha, L_g^2(0,1))} \leq C_F.$$

(d) $\tilde{F}_\varepsilon$, $\tilde{F}_0^\varepsilon$ *and* $\tilde{F}_0$ *have an uniform bounded support in* $\varepsilon \geq 0$*, that is:*

$$Supp\tilde{F}_\varepsilon \subset \{u \in X_\varepsilon^\alpha : \|u\|_{X_\varepsilon^\alpha} < 2R\},$$

$$Supp\tilde{F}_0^\varepsilon \subset \{u \in X_0^\alpha : \|Eu\|_{X_\varepsilon^\alpha} < 2R\},$$

$$Supp\tilde{F}_0 \subset \{u \in X_0^\alpha : \|u\|_{X_0^\alpha} < 2R\},$$

(e) For all $u \in X_0^\alpha$*,*

$$E\tilde{F}_0^\varepsilon(u) = \tilde{F}_\varepsilon(Eu), \qquad and \quad ED\tilde{F}_0^\varepsilon(u) = D\tilde{F}_\varepsilon(Eu)E. \tag{3.3.14}$$

and, for any compact set $K \subset X_0^\alpha$*, we have,*

$$\sup_{u_0 \in K} \|\tilde{F}_\varepsilon(Eu_0) - E\tilde{F}_0(u_0)\|_{X_\varepsilon^\alpha} \to 0, \tag{3.3.15}$$

$$\sup_{u_0 \in K} \|\tilde{F}_0^\varepsilon(u_0) - \tilde{F}_0(u_0)\|_{X_0^\alpha} \to 0, \tag{3.3.16}$$

as $\varepsilon \to 0$*.*

Remark 3.3.5. *In particular, hypothesis* **(H2')** *hold for the three nonlinearities,* $\tilde{F}_\varepsilon$, $\tilde{F}_0^\varepsilon$ *and* $\tilde{F}_0$*. Moreover, the value of* $\rho(\varepsilon)$ *and* $\beta(\varepsilon)$ *from* (2.1.15) *and* (2.2.2)*, which depend on the nonlinearities we are considering, are the following:*

$$\rho(\varepsilon), \beta(\varepsilon) = \begin{cases} 0 & \text{with the nonlinearities } \tilde{F}_\varepsilon \text{ and } \tilde{F}_0^\varepsilon \\ o(1) & \text{with the nonlinearities } \tilde{F}_\varepsilon \text{ and } \tilde{F}_0 \\ o(1) & \text{with the nonlinearities } \tilde{F}_0^\varepsilon \text{ and } \tilde{F}_0 \end{cases}$$

Proof. (a) This follows directly from definition of $\tilde{F}_\varepsilon$, $\tilde{F}_0^\varepsilon$ and $\tilde{F}_0$, see (3.3.5)-(3.3.7).
(b) We proceed as follows. Since F_ε and Θ_ε are globally Lipschitz from X_ε^α to $L^2(Q)$, $\varepsilon > 0$, and from X_0^α to $L_g^2(0,1)$ see Lemma 3.3.1 and [49], Lemma 15.7, then F_ε, $\tilde{F}_0^\varepsilon$, $\tilde{F}_0$, are globally Lipschitz from X_ε^α to $L^2(Q)$ and from X_0^α to $L_g^2(0,1)$, respectively. So, it remains to prove estimate 3.3.11.

Note that, $D\tilde{F}_\varepsilon(u) = \Theta_\varepsilon(u)DF_\varepsilon(u) + F_\varepsilon(u)D\Theta_\varepsilon(u)$. Then, we can decompose $\|D\tilde{F}_\varepsilon(u) - D\tilde{F}_\varepsilon(v)\|_{\mathcal{L}(X_\varepsilon^\alpha, L^2(Q))}$ as follows,

$$\|D\tilde{F}_\varepsilon(u) - D\tilde{F}_\varepsilon(v)\|_{\mathcal{L}(X_\varepsilon^\alpha, L^2(Q))} \leq$$

$$\|[\Theta_\varepsilon(u) - \Theta_\varepsilon(v)]DF_\varepsilon(u)\|_{\mathcal{L}(X_\varepsilon^\alpha, L^2(Q))} + \|\Theta_\varepsilon(v)[DF_\varepsilon(u) - DF_\varepsilon(v)]\|_{\mathcal{L}(X_\varepsilon^\alpha, L^2(Q))} +$$

$$+ \|[F_\varepsilon(u) - F_\varepsilon(v)]D\Theta_\varepsilon(u)\|_{\mathcal{L}(X_\varepsilon^\alpha, L^2(Q))} + \|F_\varepsilon(v)[D\Theta_\varepsilon(u) - D\Theta_\varepsilon(v)]\|_{\mathcal{L}(X_\varepsilon^\alpha, L^2(Q))} =$$

$$= I_1 + I_2 + I_3 + I_4.$$

Since Θ_ε is globally Lipschitz with uniform Lipschitz constant, that we denote by $L_{\hat{\Theta}}$, see [49], Lemma 15.7, and $\|DF_\varepsilon(u)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} \leq L_F$, see Lemma 3.3.1, then

$$I_1 \leq L_{\hat{\Theta}} L_F \|u - v\|_{X^\alpha_\varepsilon}.$$

Moreover, by Lemma 3.3.1 $F_\varepsilon \in C^{1,\theta_F}(X^\alpha_\varepsilon, L^2(Q))$. Hence,

$$I_2 \leq L_F \|u - v\|^{\theta_F}_{X^\alpha_\varepsilon}, \qquad \text{and} \qquad I_3 \leq L_F L_{\hat{\Theta}} \|u - v\|_{X^\alpha_\varepsilon}.$$

To obtain an estimate for I_4, we first calculate the expression for $D\Theta_\varepsilon(u)$. By definition of Θ_ε, see (3.3.4), we have for any $u \in X^\alpha_\varepsilon$,

$$D\Theta_\varepsilon(u) = \hat{\Theta}'(\|u\|^2_{X^\alpha_\varepsilon}) 2(u, \cdot)_{X^\alpha_\varepsilon},$$

where the function $\hat{\Theta}$ is defined in (3.3.4), $'$ is the usual derivative and $(\cdot, \cdot)_{X^\alpha_\varepsilon}$ is the scalar product in the Hilbert space X^α_ε. Hence,

$$I_4 \leq C_F \sup_{\|\varphi\|_{X^\alpha_\varepsilon}=1} \left\{ \left| \hat{\Theta}'(\|u\|^2_{X^\alpha_\varepsilon}) 2(u,\varphi)_{X^\alpha_\varepsilon} - \hat{\Theta}'(\|v\|^2_{X^\alpha_\varepsilon}) 2(v,\varphi)_{X^\alpha_\varepsilon} \right| \right\}$$

where C_F is the bound from Lemma 3.3.1 i). But,

$$\begin{aligned} &\left| \hat{\Theta}'(\|u\|^2_{X^\alpha_\varepsilon}) 2(u,\varphi)_{X^\alpha_\varepsilon} - \hat{\Theta}'(\|v\|^2_{X^\alpha_\varepsilon}) 2(v,\varphi)_{X^\alpha_\varepsilon} \right| \leq \\ &\leq \left| \left(\hat{\Theta}'(\|u\|^2_{X^\alpha_\varepsilon}) - \hat{\Theta}'(\|v\|^2_{X^\alpha_\varepsilon}) \right) 2(u,\varphi)_{X^\alpha_\varepsilon} \right| + \left| \Theta'(\|v\|^2_{X^\alpha_\varepsilon}) 2(u - v, \varphi) \right| = \\ &= I_{41} + I_{42}. \end{aligned}$$

We first analyze I_{41}. Since $\hat{\Theta}$ is a C^∞ function with bounded support in $\mathbb{R}$, then $\hat{\Theta}'$ is globally Lipschitz with Lipschitz constant $L_{\hat{\Theta}}$. So,

$$\begin{aligned} I_{41} &\leq 2L_{\hat{\Theta}} \|u\|_{X^\alpha_\varepsilon} \|\varphi\|_{X^\alpha_\varepsilon} \left| \|u\|^2_{X^\alpha_\varepsilon} - \|v\|^2_{X^\alpha_\varepsilon} \right| = \\ &= 2L_{\hat{\Theta}} \|u\|_{X^\alpha_\varepsilon} \left| (\|u\|_{X^\alpha_\varepsilon} + \|v\|_{X^\alpha_\varepsilon})(\|u\|_{X^\alpha_\varepsilon} - \|v\|_{X^\alpha_\varepsilon}) \right| \leq \\ &\leq 2L_{\hat{\Theta}} \|u\|_{X^\alpha_\varepsilon} \left(\|u\|_{X^\alpha_\varepsilon} + \|v\|_{X^\alpha_\varepsilon} \right) \|u - v\|_{X^\alpha_\varepsilon}. \end{aligned}$$

We distinguish the following cases:

(1) If $\|u\|^2_{X^\alpha_\varepsilon}, \|v\|^2_{X^\alpha_\varepsilon} \leq 8R^2$, then

$$I_{41} \leq 32 L_{\hat{\Theta}} R^2 \|u - v\|_{X^\alpha_\varepsilon}.$$

(2) If $\|u\|^2_{X^\alpha_\varepsilon}, \|v\|^2_{X^\alpha_\varepsilon} \geq 8R^2$, then $I_{41} = 0$, beacause $\Theta'(\|u\|^2_{X^\alpha_\varepsilon}) = \Theta'(\|v\|^2_{X^\alpha_\varepsilon}) = 0$

(3) If $\|u\|^2_{X^\alpha_\varepsilon} \leq 8R^2$ and $\|v\|^2_{X^\alpha_\varepsilon} \geq 8R^2$, then we always have $\Theta'(\|v\|^2_{X^\alpha_\varepsilon}) = 0$. We also distinguish two cases,

(3.1) If $\|u\|^2_{X^\alpha_\varepsilon} \geq 4R^2$, then again $\Theta'(\|u\|^2_{X^\alpha_\varepsilon}) = 0$ and therefore $I_{41} = 0$.

(3.2) If $\|u\|^2_{X^\alpha_\varepsilon} \leq 4R^2$, then $\|u - v\|_{X^\alpha_\varepsilon} \geq |\|u\|_{X^\alpha_\varepsilon} - \|v\|_{X^\alpha_\varepsilon}| \geq \frac{1}{2}R$. So, $1 \leq \frac{2}{R}\|u - v\|_{X^\alpha_\varepsilon}$, and

$$I_{41} \leq 8R^2|\Theta'(\|u\|^2_{X^\alpha_\varepsilon})| \leq 16R^2 L_{\hat{\Theta}} \frac{\|u - v\|_{X^\alpha_\varepsilon}}{R} = 16RL_{\hat{\Theta}}\|u - v\|_{X^\alpha_\varepsilon}$$

Therefore,

$$I_{41} \leq 32L_{\hat{\Theta}}R^2\|u - v\|_{X^\alpha_\varepsilon}.$$

Term I_{42} can be directly estimated as follows,

$$I_{42} \leq 2L_{\hat{\Theta}}\|u - v\|_{X^\alpha_\varepsilon}.$$

So

$$I_4 \leq (32R^2 + 2)L_{\hat{\Theta}}\|u - v\|_{X^\alpha_\varepsilon}.$$

Hence, putting all the information together, we get

$$\|D\tilde{F}_\varepsilon(u) - D\tilde{F}_\varepsilon(v)\|_{\mathcal{L}(X^\alpha_\varepsilon, L^2(Q))} \leq L_F\|u - v\|^{\theta_F}_{X^\alpha_\varepsilon},$$

with $L_F > 0$ independent of ε, as we wanted to prove.

To obtain the same result for $\tilde{F}^\varepsilon_0$ and $\tilde{F}_0$, the proof is exactly the same, step by step.

(c) This property follows from Lemma 3.3.1, item (i).

(d) It follows directly from the definition of Θ_ε and Θ_0.

(e) Finally, note that $F_0(u) = f(u(x)) = F_\varepsilon(Eu)$. Then, for $u \in X^\alpha_0$,

$$E\tilde{F}^\varepsilon_0(u) = \Theta^R_\varepsilon(Eu)EF_0(u) = \Theta^R_\varepsilon(Eu)f(u(x)) = \Theta^R_\varepsilon(Eu)F_\varepsilon(Eu) = \tilde{F}_\varepsilon(Eu),$$

and, since $D\tilde{F}^\varepsilon_0(u) = \Theta^R_\varepsilon(Eu)DF_0(u) + F_0(u)D\Theta^R_\varepsilon(Eu)$,

$$ED\tilde{F}^\varepsilon_0(u) = \Theta^R_\varepsilon(Eu)EDF_0(u) + EF_0(u)D\Theta^R_\varepsilon(Eu) =$$

$$= \Theta^R_\varepsilon(Eu)DF_\varepsilon(Eu)E + F_\varepsilon(Eu)D\Theta^R_\varepsilon(Eu) = D\tilde{F}_\varepsilon(Eu)E.$$

Moreover, for any $u_0 \in K \subset X^\alpha_0$ with K compact, we have,

$$\begin{gathered}
\|\tilde{F}_\varepsilon(Eu_0) - E\tilde{F}_0(u_0)\|_{X_\varepsilon} \leq \\
\|[\Theta^R_\varepsilon(Eu_0) - \Theta^R_0(u_0)]F_\varepsilon(Eu_0)\|_{X_\varepsilon} + \|\Theta^R_0(u_0)[F_\varepsilon(Eu_0) - EF_0(u_0)]\|_{X_\varepsilon} = \\
\|[\Theta^R_\varepsilon(Eu_0) - \Theta^R_0(u_0)]F_\varepsilon(Eu_0)\|_{X_\varepsilon} \leq C_F L_{\hat{\Theta}}|\|Eu_0\|^2_{X^\alpha_\varepsilon} - \|u_0\|^2_{X^\alpha_0}| = \\
C_F L_{\hat{\Theta}}|(\|Eu_0\|_{X^\alpha_\varepsilon} + \|u_0\|_{X^\alpha_0})(\|Eu_0\|_{X^\alpha_\varepsilon} - \|u_0\|_{X^\alpha_0})| \leq \\
C_F L_{\hat{\Theta}}(2e^2 + 1)\|u_0\|_{X^\alpha_0}|\|Eu_0\|_{X^\alpha_\varepsilon} - \|u_0\|_{X^\alpha_0}|,
\end{gathered}$$

in the last inequality we have applied the bound for operator E obtained in (3.1.19).

Hence, since K is a compact subset of X_0^α, by Lemma 3.1.3 item (iv),

$$\sup_{u_0 \in K} \|\tilde{F}_\varepsilon(Eu_0) - E\tilde{F}_0(u_0)\|_{X_\varepsilon} \to 0,$$

when ε tends to zero.

We omit the proof of (3.3.16) for being equal to the proof of (3.3.15).

■

3.4. Inertial manifolds and reduced systems

We present the construction of inertial manifolds for problems (3.3.8), (3.3.9) and (3.3.10). Remember that, with these manifolds, our problem is reduced to analyze what happens in $\mathbb{R}^m$. For this, we also study the convergence of the reduced systems in $\mathbb{R}^m$ and calculate an estimate for the distance of the time one maps corresponding to the finite dimensional systems associated to (3.3.8) and (3.3.9).

The existence of these manifolds is guaranteed by the existence of spectral gaps, large enough, in the spectrum of the associated linear elliptic operators, see [52]. Hence, we now focus on the elliptic part.

Note that both elliptic problems (3.2.1) and (3.2.2) can be written as abstract elliptic problems of the form,

$$A_\varepsilon u_\varepsilon = f_\varepsilon \qquad \text{and} \qquad A_0 v_\varepsilon = h_\varepsilon,$$

where the operators A_ε and A_0 are defined as

$$A_\varepsilon : D(A_\varepsilon) \subset L^2(Q) \to L^2(Q),$$

with

$$A_\varepsilon = -\frac{\partial^2}{\partial x^2} - \frac{1}{\varepsilon^2}\Delta_y + \alpha I, \qquad \text{and} \qquad D(A_\varepsilon) = \{u \in H^2(Q) : \frac{\partial u}{\partial \nu} = 0, \text{ at } \partial Q\}$$

and

$$A_0 : D(A_0) \subset L^2_g(0,1) \to L^2_g(0,1),$$

with

$$A_0 v = -\frac{1}{g}(g v_x)_x + \alpha v \qquad \text{and} \qquad D(A_0) = \{v \in H^2_g(0,1), u'(0) = u'(1) = 0\}.$$

By Proposition 3.2.1 for $h_\varepsilon = M f_\varepsilon$ we have

$$\|A_\varepsilon^{-1} - E A_0^{-1} M\|_{\mathcal{L}(L^2(Q), H^1_\varepsilon(Q))} \leq C\varepsilon, \tag{3.4.1}$$

and for $f_\varepsilon = Eh_\varepsilon$, see Remark 3.2.2,

$$\|A_\varepsilon^{-1}E - EA_0^{-1}\|_{\mathcal{L}(L^2_g(0,1),H^1_\varepsilon(Q))} \leq C\varepsilon. \tag{3.4.2}$$

The limit operator A_0 is of Sturm-Liouville type of one dimension. Following [31], Lemma 4.2, we know that there exists N_0 such that for all $m \geq N_0$

$$\pi^2\left(m+\frac{1}{4}\right)^2 \leq \lambda_m^0 \leq \pi^2\left(m+\frac{3}{4}\right)^2. \tag{3.4.3}$$

This implies that for $m \geq N_0$,

$$\pi^2(m+1) \leq \lambda_{m+1}^0 - \lambda_m^0 \leq 3\pi^2(m+1).$$

So, with estimates (3.4.1) and (3.4.2) of the distance of the resolvent operators obtained in section 3.2, taking m large enough and applying the spectral continuity result obtained in previous chapter, the limit and perturbed problems satisfy the gap condition which ensures the existence of inertial manifolds in X_ε^α, $\varepsilon \geq 0$, for $0 \leq \alpha < \frac{1}{2}$, see [52]. Notice that, applying (3.4.3), for $\alpha = \frac{1}{2}$ the gap condition (2.1.13) is not satisfied.

Then, we can consider the orthogonal projections onto the spaces generated by the first m eigenfunctions, $0 \leq \varepsilon \leq \varepsilon_0$. Remember we denote by $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}$ the canonical orthogonal projection onto the eigenfunctions, $\{\varphi_i^\varepsilon\}_{i=1}^m$, corresponding to the first m eigenvalues of the operator A_ε, $0 \leq \varepsilon \leq \varepsilon_0$ and $\mathbf{Q}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}$ its orthogonal complement,

$$\begin{aligned} \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}} : L^2(Q) &\longrightarrow L^2(Q) \\ v &\longrightarrow \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(v) = \textstyle\sum_{i=1}^m (v,\varphi_i^\varepsilon)_{L^2(Q)}\varphi_i^\varepsilon \end{aligned} \tag{3.4.4}$$

or if $\varepsilon = 0$,

$$\begin{aligned} \mathbf{P}^{\mathbf{0}}_{\mathbf{m}} : L^2_g(0,1) &\longrightarrow L^2_g(0,1) \\ v &\longrightarrow \mathbf{P}^{\mathbf{0}}_{\mathbf{m}}(v) = \textstyle\sum_{i=1}^m (v,\varphi_i^0)_{L^2_g(0,1)}\varphi_i^0 \end{aligned} \tag{3.4.5}$$

As we have done in the previous chapter, we express any element belonging to the linear subspace $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(L^2(Q))$ in the following basis,

$$\{\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(E\varphi_1^0), \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(E\varphi_2^0), ..., \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(E\varphi_m^0)\}, \qquad \text{for} \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

with $\{\varphi_i^0\}_{i=1}^m$ the eigenfunctions related to the first m eigenvalues of A_0, which, as we have shown in the previous chapter, is a basis in $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(L^2(Q))$ and in $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(H^1_{\boldsymbol{\varepsilon}}(Q))$.

Let us denote by j_ε the isomorphism from $\mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(L^2(Q)) = [\psi_1^\varepsilon, ..., \psi_m^\varepsilon]$ onto $\mathbb{R}^m$, with $\psi_i^\varepsilon = \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(E\varphi_i^0)$, that gives us the coordinates of each vector. That is,

$$\begin{aligned} j_\varepsilon : \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}(L^2(Q)) &\longrightarrow \mathbb{R}^m, \\ w_\varepsilon &\longmapsto z, \end{aligned} \tag{3.4.6}$$

where $w_\varepsilon = \sum_{i=1}^m z_i \psi_i^\varepsilon$ and $z = (z_1, ..., z_m)$.

Since the gap conditions for $0 \leq \alpha < \frac{1}{2}$,

$$\lambda_{m+1}^0 - \lambda_m^0 \geq 3(\kappa + 2) L_F [(\lambda_{m+1}^0)^\alpha + (\lambda_m^0)^\alpha]$$

$$(\lambda_m^0)^{1-\alpha} \geq 6(\kappa + 2) L_F (1 - \alpha)^{-1}, \tag{3.4.7}$$

are satisfied, then, applying Proposition 2.1.2 of Chapter 2, there exist $L < 1$ and $0 < \varepsilon_1 \leq \varepsilon_0$ such that for all $0 < \varepsilon \leq \varepsilon_1$ there exist inertial manifolds $\mathcal{M}_\varepsilon$, $\mathcal{M}_0^\varepsilon$ and $\mathcal{M}_0$ for (3.3.8), (3.3.9) and (3.3.10), given by the "graph" of functions $\Phi_\varepsilon, \Phi_0^\varepsilon, \Phi_0 \in \mathcal{F}_\varepsilon(L, 2R)$,

$$\mathcal{M}_\varepsilon = \{j_\varepsilon^{-1}(z) + \Phi_\varepsilon(z);\ z \in \mathbb{R}^m\}, \tag{3.4.8}$$

$$\mathcal{M}_0^\varepsilon = \{j_0^{-1}(z) + \Phi_0^\varepsilon(z);\ z \in \mathbb{R}^m\}, \tag{3.4.9}$$

$$\mathcal{M}_0 = \{j_0^{-1}(z) + \Phi_0(z);\ z \in \mathbb{R}^m\}, \tag{3.4.10}$$

where, for the α fixed in the previous section, $0 < \alpha < \frac{1}{2}$,

$$\mathcal{F}_\varepsilon(L, 2R) = \{\Phi_\varepsilon : \mathbb{R}^m \to X_\varepsilon^\alpha, \text{ such that } \|\Phi_\varepsilon(z) - \Phi_\varepsilon(z')\|_{X_\varepsilon^\alpha} \leq L|z - z'|_{\varepsilon,\alpha}, \quad z, z' \in \mathbb{R}^m,$$
$$\text{and} \quad \operatorname{supp} \Phi_\varepsilon \subset B_{2R}\}.$$

If we denote by $T_{\mathcal{M}_\varepsilon}$, $T_{\mathcal{M}_0^\varepsilon}$ and $T_{\mathcal{M}_0}$ the time one maps of the semigroup restricted to the inertial manifolds $\mathcal{M}_\varepsilon$, $\mathcal{M}_0^\varepsilon$ and $\mathcal{M}_0$, respectively, for $u_\varepsilon \in \mathcal{M}_\varepsilon$, $u_0^\varepsilon \in \mathcal{M}_0^\varepsilon$ and $u_0 \in \mathcal{M}_0$ and $z \in \mathbb{R}^m$, the time one maps satisfy the following equalities,

$$T_{\mathcal{M}_\varepsilon}(u_\varepsilon) = p_\varepsilon(1) + \Phi_\varepsilon(j_\varepsilon(p_\varepsilon(1))),$$
$$T_{\mathcal{M}_0^\varepsilon}(u_0^\varepsilon) = p_0^\varepsilon(1) + \Phi_0^\varepsilon(j_0(p_0^\varepsilon(1))),$$
$$T_{\mathcal{M}_0}(u_0) = p_0(1) + \Phi_0(j_0(p_0(1))),$$

with $p_\varepsilon(t)$, $p_0^\varepsilon(t)$ and $p_0(t)$ the solutions of

$$\begin{cases} p_t = -A_\varepsilon p + \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} \tilde{F}_\varepsilon(p + \Phi_\varepsilon(j_\varepsilon(p(t)))) \\ p(0) = j_\varepsilon^{-1}(z), \end{cases} \tag{3.4.11}$$

$$\begin{cases} p_t = -A_0 p + \mathbf{P}_\mathbf{m}^\mathbf{0} \tilde{F}_0^\varepsilon(p + \Phi_0^\varepsilon(j_0(p(t)))) \\ p(0) = j_0^{-1}(z), \end{cases} \tag{3.4.12}$$

$$\begin{cases} p_t = -A_0 p + \mathbf{P}_\mathbf{m}^\mathbf{0} \tilde{F}_0(p + \Phi_0(j_0(p(t)))) \\ p(0) = j_0^{-1}(z). \end{cases} \tag{3.4.13}$$

Moreover, $j_\varepsilon(p_\varepsilon(t))$, $j_0(p_0^\varepsilon(t))$ and $j_0(p_0(t))$ satisfy the following systems in $\mathbb{R}^m$,

$$\begin{cases} z_t = -j_\varepsilon A_\varepsilon j_\varepsilon^{-1} z + j_\varepsilon \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} \tilde{F}_\varepsilon(j_\varepsilon^{-1}(z) + \Phi_\varepsilon(z)) \\ z(0) = z^0, \end{cases} \tag{3.4.14}$$

$$\begin{cases} z_t = -j_0 A_0 j_0^{-1} z + j_0 \mathbf{P}_{\mathbf{m}}^{\mathbf{0}} \tilde{F}_0^\varepsilon(j_0^{-1}(z) + \Phi_0^\varepsilon(z)) \\ z(0) = z^0, \end{cases} \tag{3.4.15}$$

$$\begin{cases} z_t = -j_0 A_0 j_0^{-1} z + j_0 \mathbf{P}_{\mathbf{m}}^{\mathbf{0}} \tilde{F}_0(j_0^{-1}(z) + \Phi_0(z)) \\ z(0) = z^0. \end{cases} \tag{3.4.16}$$

We write them in the following way:

$$\begin{cases} z_t = -j_\varepsilon A_\varepsilon j_\varepsilon^{-1} z + H_\varepsilon(z) \\ z(0) = z^0, \end{cases} \tag{3.4.17}$$

$$\begin{cases} z_t = -j_0 A_0 j_0^{-1} z + H_0^\varepsilon(z) \\ z(0) = z^0, \end{cases} \tag{3.4.18}$$

$$\begin{cases} z_t = -j_0 A_0 j_0^{-1} z + H_0(z) \\ z(0) = z^0, \end{cases} \tag{3.4.19}$$

where

$$H_\varepsilon, H_0^\varepsilon, H_0 : \mathbb{R}^m \longrightarrow \mathbb{R}^m,$$

are given by

$$H_\varepsilon = j_\varepsilon \mathbf{P}_{\mathbf{m}}^{\boldsymbol{\varepsilon}} \tilde{F}_\varepsilon(j_\varepsilon^{-1}(z) + \Phi_\varepsilon(z)),$$

$$H_0^\varepsilon = j_0 \mathbf{P}_{\mathbf{m}}^{\mathbf{0}} \tilde{F}_0^\varepsilon(j_0^{-1}(z) + \Phi_0^\varepsilon(z)),$$

$$H_0 = j_0 \mathbf{P}_{\mathbf{m}}^{\mathbf{0}} \tilde{F}_0(j_0^{-1}(z) + \Phi_0(z)).$$

They are of compact support,

$$supp(H_\varepsilon), supp(H_0^\varepsilon), supp(H_0) \subset B_{R'},$$

and $B_{R'}$ denotes a ball in $\mathbb{R}^m$ of some radius $R' > 0$ centered at the origin.

Let us denote by $\bar{T}_\varepsilon, \bar{T}_0^\varepsilon, \bar{T}_0 : \mathbb{R}^m \to \mathbb{R}^m$, the time one maps of the dynamical systems generated by (3.4.17), (3.4.18) and (3.4.19), respectively.

We have the following result

Proposition 3.4.1. *If all equilibria of* (3.1.7) *are hyperbolic, then the time one map of* (3.4.19) *is a Morse-Smale (gradient like) map.*

Proof. Since all the equilibrium points of (3.1.7) are hyperbolic, by [33] the stable and unstable manifolds intersect transversally and so, the time one map of the dynamical system generated by (3.3.10) is a Morse-Smale (gradient like) map. In [42], Section 3.4, S. Y. Pilyugin proves that, then, the time one map $T_{\mathcal{M}_0}$ corresponding to the limit system in the inertial manifold $\mathcal{M}_0$ is a Morse-Smale (gradient like) map in a neighborhood V of the attractor $\mathcal{A}_0$ in this inertial manifold, $V \subset \mathcal{M}_0$. Then, the time one map $\bar{T}_0$ of the limit system in $\mathbb{R}^m$ generated by (3.4.19) is Morse-Smale (gradient like).

■

Remark 3.4.2. *The fact that all equilibria is hyperbolic is a generic situation.*

3.5. Rate of the distance of attractors

In this section we give an estimate for the distance of attractors related to (3.1.3) and (3.1.7), proving our main result, Theorem 3.1.2. To accomplish this, we start showing the following important results about the relation of the time one maps of the dynamical systems related to (3.4.17), (3.4.18) and (3.4.19) and the ones corresponding to (3.1.7) and (3.1.11).

We analyze its convergence.

Lemma 3.5.1. *We have,*

$$\|\bar{T}_\varepsilon - \bar{T}_0^\varepsilon\|_{C^1(\mathbb{R}^m,\mathbb{R}^m)} \to 0,$$

$$\|\bar{T}_0^\varepsilon - \bar{T}_0\|_{C^1(\mathbb{R}^m,\mathbb{R}^m)} \to 0,$$

as $\varepsilon \to 0$. Moreover, we have,

$$\|\bar{T}_\varepsilon - \bar{T}_0^\varepsilon\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^m)} \le C\varepsilon|\log(\varepsilon)|, \tag{3.5.1}$$

with C independent of ε.

Proof. Note that $\tilde{F}_\varepsilon \in C^{1,\theta_F}(X_\varepsilon^\alpha, L^2(Q))$, $\tilde{F}_0^\varepsilon, \tilde{F}_0 \in C^{1,\theta_F}(X_0^\alpha, L^2_g(0,1))$, see Lemma 3.3.4 item (b), and $\Phi_\varepsilon \in C^{1,\theta}(\mathbb{R}^m, X_\varepsilon^\alpha)$, $\Phi_0^\varepsilon, \Phi_0 \in C^{1,\theta}(\mathbb{R}^m, X_0^\alpha)$ for certain small θ, see Proposition 2.2.1 of Chapter 2. Then, it is easy to show that $H_\varepsilon, H_0^\varepsilon, H_0 \in C^{1,\theta}(\mathbb{R}^m,\mathbb{R}^m)$ for $\theta > 0$ small enough and

$$\|H_\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m,\mathbb{R}^m)}, \|H_0^\varepsilon\|_{C^{1,\theta}(\mathbb{R}^m,\mathbb{R}^m)}, \|H_0\|_{C^{1,\theta}(\mathbb{R}^m,\mathbb{R}^m)} \le \mathbf{M}, \tag{3.5.2}$$

with $\mathbf{M}$ independent of ε. Moreover, by Lemma 3.3.4 item (e) we have that,

$$\|\tilde{F}_\varepsilon E - E\tilde{F}_0^\varepsilon\|_{C^0(X_0^\alpha, X_\varepsilon)} = 0,$$

and for $K = \{u_0 = p_0 + \Phi_0(p_0)$ with $p_0 \in [\varphi_1^0, ..., \varphi_m^0]$ and $\|p_0\|_{X_0^\alpha} \leq 2R\} \subset X_0^\alpha$

$$\sup_{u_0 \in K} \|\tilde{F}_0^\varepsilon(u_0) - \tilde{F}_0(u_0)\|_{X_0} \to 0,$$

as $\varepsilon \to 0$. Then, since we have $j_\varepsilon \to j_0$ and $\mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}} \to \mathbf{P}_\mathbf{m}^\mathbf{0}$, see Remark 2.1.8 and Lemma 2.1.12 of Chapter 2, we have that

$$\|H_\varepsilon - H_0^\varepsilon\|_{C^0(\mathbb{R}^m,\mathbb{R}^m)} \to 0, \qquad \|H_0^\varepsilon - H_0\|_{C^0(\mathbb{R}^m,\mathbb{R}^m)} \to 0. \tag{3.5.3}$$

Hence, (3.5.2), (3.5.3) and the fact that the support is contained in $B_{R'}$ imply

$$\|H_\varepsilon - H_0^\varepsilon\|_{C^{1,\theta'}(B_{R'},\mathbb{R}^m)} \to 0, \qquad \|H_0^\varepsilon - H_0\|_{C^{1,\theta'}(B_{R'},\mathbb{R}^m)} \to 0,$$

as $\varepsilon \to 0$, for $\theta' < \theta$. For this, we are using the compact embedding $C^{1,\theta}(B,\mathbb{R}^m) \hookrightarrow C^{1,\theta'}(B,\mathbb{R}^m)$ for all $\theta' < \theta$, the convergence (3.5.3) and the boundness of $H_\varepsilon, H_0^\varepsilon, H_0$ in $C^{1,\theta}(B,\mathbb{R}^m)$. In particular, we have this convergence in the C^1-topology.

With this, we obtain the desired convergence,

$$\|\bar{T}_\varepsilon - \bar{T}_0^\varepsilon\|_{C^1(\mathbb{R}^m,\mathbb{R}^m)} \to 0,$$
$$\|\bar{T}_0^\varepsilon - \bar{T}_0\|_{C^1(\mathbb{R}^m,\mathbb{R}^m)} \to 0.$$

Now, since systems (3.3.8) and (3.3.9) satisfy hypotheses **(H1)** and **(H2')** required in the previous chapter, then, we can apply all the results obtained in Chapter 2 to obtain estimate 3.5.1. Hence,

$$\|\bar{T}_0^\varepsilon - \bar{T}_\varepsilon\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^m)} = \sup_{z\in\mathbb{R}^m} |\bar{T}_0^\varepsilon(z) - \bar{T}_\varepsilon(z)|_{0,\alpha} =$$
$$\sup_{z\in\mathbb{R}^m} |z_0^\varepsilon(1) - z_\varepsilon(1)|_{0,\alpha} = \sup_{z\in\mathbb{R}^m} |j_0(p_0^\varepsilon(1)) - j_\varepsilon(p_\varepsilon(1))|_{0,\alpha},$$

where $p_\varepsilon(t)$ and $p_0^\varepsilon(t)$ are the solutions of (3.4.11) and (3.4.12) with $p_\varepsilon(0) = j_\varepsilon^{-1}(z)$, $p_0^\varepsilon(0) = j_0^{-1}(z)$, and z_ε, z_0^ε, the solutions of (3.4.14) and (3.4.15) with $z_\varepsilon(0) = z$, $z_0^\varepsilon(0) = z$.

By Lemma 2.1.21 and since $\kappa = 2e^2$, we obtain,

$$|j_0(p_0^\varepsilon(1)) - j_\varepsilon(p_\varepsilon(1))|_{0,\alpha} \leq (2e^2+1)\|Ep_0^\varepsilon(1) - p_\varepsilon(1)\|_{X_\varepsilon^\alpha} + (2e^2+1)C_P\varepsilon\|p_0^\varepsilon(1)\|_{L_g^2(0,1)},$$

with $C_P \sim (\lambda_m^0)^3$ a constant from the estimate of the distance of spectral projections, $\|E\mathbf{P}_\mathbf{m}^\mathbf{0} - \mathbf{P}_\mathbf{m}^{\boldsymbol{\varepsilon}}E\|_{\mathcal{L}(L_g^2(0,1),X_\varepsilon^\alpha)}$, see Lemma 2.1.12.

Moreover, since $\|E\tilde{F}_0^\varepsilon - \tilde{F}_\varepsilon E\|_{L^\infty(X_0^\alpha,L^2(Q))} = 0$, (see Lemma 3.3.4, item (e)) applying Corollaly 2.1.24 of Chapter 2 and Proposition 3.2.1 we have

$$\|Ep_0^\varepsilon(1) - p_\varepsilon(1)\|_{X_\varepsilon^\alpha} \leq C(\|E\Phi_0^\varepsilon - \Phi_\varepsilon\|_{L^\infty(\mathbb{R}^m,X_\varepsilon^\alpha)} + \varepsilon).$$

Then,

$$\|\bar{T}_0^\varepsilon - \bar{T}_\varepsilon\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^m)} = \sup_{z\in\mathbb{R}^m} |\bar{T}_0^\varepsilon(z) - \bar{T}_\varepsilon(z)|_{0,\alpha} \leq$$

$$\leq C(\|E\Phi_0^\varepsilon - \Phi_\varepsilon\|_{L^\infty(\mathbb{R}^m, X_\varepsilon^\alpha)} + \varepsilon) \leq C\varepsilon|\log(\varepsilon)|, \tag{3.5.4}$$

with $C > 0$ independent of ε. Last inequality is obtained applying Theorem 2.1.4, Proposition 3.2.1 and Lemma 3.3.4, item (e).

■

Remark 3.5.2. *Note that an estimate for the rate of convergence of* $\|\bar{T}_0 - \bar{T}_\varepsilon\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^m)}$ *and* $\|\bar{T}_0^\varepsilon - \bar{T}_0\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^m)}$ *is not obtained in a straightforward way. More precisely, the difficulty lies in analyzing the rate of convergence of* $\|Eu_0\|_{X_\varepsilon^\alpha} \to \|u_0\|_{X_0^\alpha}$*, see Lemma 3.1.3.*

We now give an estimate for the distance of the time one maps of the dynamical systems generated by (3.1.7) and (3.1.11)

Lemma 3.5.3. *Let* T_0 *and* T_ε*,* $0 < \varepsilon \leq \varepsilon_0$*, the time one maps corresponding to (3.1.7) and (3.1.11), respectively. Then, for* $R > 0$ *large enough, there exists a constant* $C = C(R)$ *such that for any* $w_0 \in L_g^2(0,1)$*, with* $\|w_0\|_{L_g^2(0,1)} \leq R$*, we have,*

$$\|T_\varepsilon(Ew_0) - ET_0(w_0)\|_{H_\varepsilon^1(Q)} \leq C\varepsilon|\log(\varepsilon)|.$$

Proof. We have denoted previously by $S_\varepsilon(t)$ and $S_0(t)$ the nonlinear semigroups generated by (3.1.11) and (3.1.7) respectively, so that $T_\varepsilon = S_\varepsilon(1)$ and $T_0 = S_0(1)$. Hence, with the variation of constants formula, for $0 < t \leq 1$,

$$\|S_\varepsilon(t)(Ew_0) - ES_0(t)(w_0)\|_{H_\varepsilon^1(Q)} \leq \|(e^{-A_\varepsilon t}E - Ee^{-A_0 t})w_0\|_{H_\varepsilon^1(Q)} +$$
$$+ \int_0^t \left\| e^{-A_\varepsilon(t-s)} F_\varepsilon(S_\varepsilon(s)Ew_0) - Ee^{-A_0(t-s)} F_0(S_0(s)w_0) \right\|_{H_\varepsilon^1(Q)} ds \leq$$
$$\leq \|(e^{-A_\varepsilon t}E - Ee^{-A_0 t})w_0\|_{H_\varepsilon^1(Q)} +$$
$$+ \int_0^t \left\| \left(e^{-A_\varepsilon(t-s)}E - Ee^{-A_0(t-s)} \right) F_0(S_0(s)w_0) \right\|_{H_\varepsilon^1(Q)} ds +$$
$$+ \int_0^t \left\| e^{-A_\varepsilon(t-s)} \left(F_\varepsilon(ES_0(s)w_0) - F_0(S_0(s)w_0) \right) \right\|_{H_\varepsilon^1(Q)} ds +$$
$$+ \int_0^t \left\| e^{-A_\varepsilon(t-s)} \left(F_\varepsilon(S_\varepsilon(s)Ew_0) - F_\varepsilon(ES_0(s)w_0) \right) \right\|_{H_\varepsilon^1(Q)} ds.$$

But notice that since both F_ε and F_0 are Nemitskii operators of the same function $f : \mathbb{R} \to \mathbb{R}$ then $F_\varepsilon(ES_0(s)w_0) = F_0(S_0(s)w_0)$, and the third term is identically 0.

Now, since hypothesis **(H1)** is satisfied, applying Lemma 2.1.14, Lemma 2.1.15, Lemma 3.1.3, Proposition 3.2.1 and with Gronwall-Henry inequality, see [32] Section 7, for $t = 1$, we obtain,

$$\|T_\varepsilon(Ew_0) - ET_0(w_0)\|_{H^1_\varepsilon(Q)} = \|S_\varepsilon(1)(Ew_0) - ES_0(1)(w_0)\|_{H^1_\varepsilon(Q)} \leq C\varepsilon|\log(\varepsilon)|,$$

with $C > 0$ independent of ε.

■

We show the time one maps are Lipschitz from $L^2(Q)$ to $H^1_\varepsilon(Q)$ uniformly in ε.

Lemma 3.5.4. *There exists a constant $C > 0$ independent of ε so that, for $0 \leq \varepsilon \leq \varepsilon_0$,*

$$\|T_\varepsilon(u_\varepsilon) - T_\varepsilon(w_\varepsilon)\|_{H^1_\varepsilon(Q)} \leq C\|u_\varepsilon - w_\varepsilon\|_{L^2(Q)}.$$

Proof. By the variation of constants formula, for $0 < t \leq 1$, we have

$$\|S_\varepsilon(t)u_\varepsilon - S_\varepsilon(t)w_\varepsilon\|_{H^1_\varepsilon(Q)} \leq \|e^{-A_\varepsilon t}(u_\varepsilon - w_\varepsilon)\|_{H^1_\varepsilon(Q)} +$$

$$+ \int_0^t \left\| e^{-A_\varepsilon(t-s)}(F_\varepsilon(S_\varepsilon(s)u_\varepsilon) - F_\varepsilon(S_\varepsilon(s)w_\varepsilon)) \right\|_{H^1_\varepsilon(Q)} ds.$$

Applying Lemma 2.1.6 of Chapter 2 and Lemma 3.3.1, item (ii),

$$\|S_\varepsilon(t)u_\varepsilon - S_\varepsilon(t)w_\varepsilon\|_{H^1_\varepsilon(Q)} \leq Ce^{-\lambda_1^\varepsilon t}t^{-\frac{1}{2}}\|u_\varepsilon - w_\varepsilon\|_{L^2(Q)} +$$

$$+CL_F e^{-\lambda_1^\varepsilon t}\int_0^t e^{\lambda_1^\varepsilon s}(t-s)^{-\frac{1}{2}}\|S_\varepsilon(s)u_\varepsilon - S_\varepsilon(s)w_\varepsilon\|_{H^1_\varepsilon(Q)} ds.$$

Applying Gronwall inequality, for $0 < t \leq 1$, we have

$$\|S_\varepsilon(t)u_\varepsilon - S_\varepsilon(t)w_\varepsilon\|_{H^1_\varepsilon(Q)} \leq Ct^{-\frac{1}{2}}\|u_\varepsilon - w_\varepsilon\|_{L^2(Q)}e^{-\lambda_1^\varepsilon t},$$

with $C > 0$ independent of ε.

Then, for the time one map $T_\varepsilon = S_\varepsilon(1)$ we obtain

$$\|T_\varepsilon(u_\varepsilon) - T_\varepsilon(w_\varepsilon)\|_{H^1_\varepsilon(Q)} \leq C\|u_\varepsilon - w_\varepsilon\|_{L^2(Q)},$$

with $C > 0$ independent of ε, which shows the result.

■

We proceed to prove the main result of this chapter.

Proof of Theorem 3.1.2 We obtain now a rate of convergence of attractors $\mathcal{A}_0$ and $\mathcal{A}_\varepsilon$ of the dynamical systems generated by (3.1.7) and (3.1.11), respectively. We know that for any $u_0 \in \mathcal{A}_0$ and any $u_\varepsilon \in \mathcal{A}_\varepsilon$ there exist a $w_0 \in \mathcal{A}_0$ and $w_\varepsilon \in \mathcal{A}_\varepsilon$ such that,

$$u_0 = T_0(w_0), \qquad \text{and} \quad u_\varepsilon = T_\varepsilon(w_\varepsilon),$$

with T_0 and T_ε the time one maps corresponding to (3.1.7) and (3.1.11).

Moreover, as we have said before, for each $\varepsilon > 0$ the attractor $\mathcal{A}_\varepsilon$ is contained in the inertial manifold $\mathcal{M}_\varepsilon$ and $\mathcal{A}_0$ is contained in the inertial manifolds $\mathcal{M}_0^\varepsilon$ and $\mathcal{M}_0$. We also have that although $\mathcal{M}_\varepsilon$, $\mathcal{M}_0^\varepsilon$ and $\mathcal{M}_0$ are manifolds close enough, we only can provide explicit rates of the distance between $\mathcal{M}_\varepsilon$ and $\mathcal{M}_0^\varepsilon$ as ε goes to zero.

The Hausdorff distance of attractors $\mathcal{A}_0$ and $\mathcal{A}_\varepsilon$ in $H^1_{\boldsymbol{\varepsilon}}(Q)$, is given by

$$dist_{H^1_{\boldsymbol{\varepsilon}}(Q)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) = max\{ \sup_{u_0 \in \mathcal{A}_0} \inf_{u_\varepsilon \in \mathcal{A}_\varepsilon} \|Eu_0 - u_\varepsilon\|_{H^1_{\boldsymbol{\varepsilon}}(Q)}, \sup_{u_\varepsilon \in \mathcal{A}_\varepsilon} \inf_{u_0 \in \mathcal{A}_0} \|u_\varepsilon - Eu_0\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} \}.$$

Then, we consider $w_\varepsilon \in \mathcal{A}_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, given by $w_\varepsilon = j_\varepsilon^{-1}(z_\varepsilon) + \Phi_\varepsilon(z_\varepsilon)$ and $w_0 \in \mathcal{A}_0$, given by $w_0 = j_0^{-1}(z_0) + \Phi_0^\varepsilon(z_0)$ with $z_\varepsilon \in \bar{\mathcal{A}}_\varepsilon$ and $z_0 \in \bar{\mathcal{A}}_0$, the "projected" attractors in $\mathbb{R}^m$ corresponding to (3.4.14) and (3.4.15), respectively.

We know,

$$\|Eu_0 - u_\varepsilon\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} = \|ET_0(w_0) - T_\varepsilon(w_\varepsilon)\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} \le$$

$$\le \|ET_0(w_0) - T_\varepsilon(Ew_0)\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} + \|T_\varepsilon(Ew_0) - T_\varepsilon(w_\varepsilon)\|_{H^1_{\boldsymbol{\varepsilon}}(Q)}.$$

Applying Lemma 3.5.3 and Lemma 3.5.4, we have

$$\|Eu_0 - u_\varepsilon\|_{H^1_{\boldsymbol{\varepsilon}}(Q)} \le C\varepsilon|\log(\varepsilon)| + C\|Ew_0 - w_\varepsilon\|_{X^\alpha_\varepsilon}.$$

So, we need to estimate the norm $\|Ew_0 - w_\varepsilon\|_{X^\alpha_\varepsilon}$, where,

$$w_\varepsilon = j_\varepsilon^{-1}(z_\varepsilon) + \Phi_\varepsilon(z_\varepsilon), \qquad z_\varepsilon \in \bar{\mathcal{A}}_\varepsilon,$$

and

$$w_0 = j_0^{-1}(z_0) + \Phi_0^\varepsilon(z_0), \qquad z_0 \in \bar{\mathcal{A}}_0$$

with $\bar{\mathcal{A}}_\varepsilon$ and $\bar{\mathcal{A}}_0$ the attractors corresponding to (3.4.14) and (3.4.15).

Hence, since $j_0^{-1}(z_0) = \sum_{i=1}^m z_i^0 \psi_i^0$ and $j_\varepsilon^{-1}(z_\varepsilon) = \sum_{i=1}^m z_i^\varepsilon \psi_i^\varepsilon$,

$$\|Ew_0 - w_\varepsilon\|_{X^\alpha_\varepsilon} \le \|Ej_0^{-1}(z_0) - j_\varepsilon^{-1}(z_\varepsilon)\|_{X^\alpha_\varepsilon} + \|E\Phi_0^\varepsilon(z_0) - \Phi_\varepsilon(z_\varepsilon)\|_{X^\alpha_\varepsilon} \le$$

$$\le \|\sum_{i=1}^m (z_i^0 - z_i^\varepsilon)E\psi_i^0\|_{X^\alpha_\varepsilon} + \|\sum_{i=1}^m z_i^\varepsilon(E\psi_i^0 - \psi_i^\varepsilon)\|_{X^\alpha_\varepsilon} +$$

$$+ \|E\Phi_0^\varepsilon(z_0) - E\Phi_0^\varepsilon(z_\varepsilon)\|_{X^\alpha_\varepsilon} + \|E\Phi_0^\varepsilon(z_\varepsilon) - \Phi_\varepsilon(z_\varepsilon)\|_{X^\alpha_\varepsilon} \le$$

$$\le 4e^2|z_0 - z_\varepsilon|_{0,\alpha} + \sup_{z_\varepsilon \in \bar{\mathcal{A}}_\varepsilon} |z_\varepsilon| \|E\mathbf{P}^{\mathbf{0}}_{\mathbf{m}} - \mathbf{P}^{\boldsymbol{\varepsilon}}_{\mathbf{m}}E\|_{\mathcal{L}(L^2_g(0,1), X^\alpha_\varepsilon)} + \|E\Phi_0^\varepsilon - \Phi_\varepsilon\|_{L^\infty(\mathbb{R}^m, X^\alpha_\varepsilon)}.$$

In the last inequality we have applied the estimate of the norm of operator E, see (3.1.19).

Since $z_0 \in \bar{\mathcal{A}}_0$ and $z_\varepsilon \in \bar{\mathcal{A}}_\varepsilon$, then

$$\|Ew_0 - w_\varepsilon\|_{X^\alpha_\varepsilon} \leq 4e^2|z_0 - z_\varepsilon|_{0,\alpha} + |z_\varepsilon|\|E\mathbf{P^0_m} - \mathbf{P^\varepsilon_m}E\|_{\mathcal{L}(L^2_g(0,1),X^\alpha_\varepsilon)} +$$
$$+\|E\Phi^\varepsilon_0 - \Phi_\varepsilon\|_{L^\infty(\mathbb{R}^m,X^\alpha_\varepsilon)} = I_1 + I_2 + I_3.$$

To estimate I_2, note that we have studied the convergence of $\|E\mathbf{P^0_m}-\mathbf{P^\varepsilon_m}E\|_{\mathcal{L}(L^2_g(0,1),X^\alpha_\varepsilon)}$ in terms of the distance of the resolvent operators, see Lemma 2.1.12. Then, in our case, we have,

$$I_2 \leq C\varepsilon.$$

By Theorem 2.1.4,

$$I_3 \leq C\varepsilon|\log(\varepsilon)|.$$

Hence, putting everything together,

$$\|Ew_0 - w_\varepsilon\|_{X^\alpha_\varepsilon} \leq 4e^2|z_0 - z_\varepsilon|_{0,\alpha} + C\varepsilon|\log(\varepsilon)|,$$

with C independent of ε. Then,

$$\sup_{w_0\in\mathcal{A}_0} \inf_{w_\varepsilon\in\mathcal{A}_\varepsilon} \|Ew_0 - w_\varepsilon\|_{X^\alpha_\varepsilon} \leq 4e^2 \sup_{z_0\in\bar{\mathcal{A}}_0} \inf_{z_\varepsilon\in\bar{\mathcal{A}}_\varepsilon} |z_0 - z_\varepsilon|_{0,\alpha} + C\varepsilon|\log(\varepsilon)|.$$

Hence,

$$dist_{H^1_\varepsilon(Q)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq 4e^2 dist_{\mathbb{R}^m}(\bar{\mathcal{A}}_0, \bar{\mathcal{A}}_\varepsilon) + C\varepsilon|\log(\varepsilon)|.$$

To estimate $dist_H(\bar{\mathcal{A}}_0, \bar{\mathcal{A}}_\varepsilon)$, we need to apply techniques of Shadowing Theory described in Chapter 1. First, we have by Proposition 3.4.1, that the time one map of the system given by the ordinary differential equation (3.4.19) is a Morse-Smale map. Moreover, by Lemma 3.5.1, we can take ε small enough so that the time one maps corresponding to (3.4.17) and (3.4.18), $\bar{T}_\varepsilon$ and $\bar{T}^\varepsilon_0$, respectivelly belong to a C^1 neighborhood of $\bar{T}_0$. Then, by Chapter 1, Proposition 1.1.20

$$dist_{\mathbb{R}^m}(\bar{\mathcal{A}}_0, \bar{\mathcal{A}}_\varepsilon) \leq L\|\bar{T}^\varepsilon_0 - \bar{T}_\varepsilon\|_{L^\infty(\mathbb{R}^m,\mathbb{R}^m)},$$

with $L > 0$ independent of ε, see Proposition 1.1.20. Hence, using the estimate obtained in Lemma 3.5.1,

$$4e^2 dist_{\mathbb{R}^m}(\bar{\mathcal{A}}_0, \bar{\mathcal{A}}_\varepsilon) \leq C\varepsilon|\log(\varepsilon)|,$$

with $C > 0$ independent of ε.

Putting all together,

$$dist_{H^1_\varepsilon(Q)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq C\varepsilon|\log(\varepsilon)|,$$

with C independent of ε.

Finally, applying identity 3.1.13, we have,

$$dist_{H^1(Q_\varepsilon)}(\mathcal{A}_0, \mathcal{A}_\varepsilon) = \varepsilon^{\frac{d-1}{2}} dist_H(\mathcal{A}_0, \mathcal{A}_\varepsilon) \leq C\varepsilon^{\frac{d+1}{2}}|\log(\varepsilon)|,$$

with C independent of ε.

■

3.6. Appendix

In this section, we study the embedding of the fractional power space X_ε^α, with $0<\alpha<\frac{1}{2}$, into the Lebesgue space L^p, for an appropriate p. Remember that X_ε^α is the fractional power space corresponding to the elliptic operator A_ε, $\varepsilon \geq 0$, described in Section 3.4. Then, we have the following result.

Lemma 3.6.1. *Let X_ε^α, with $0<\alpha<\frac{1}{2}$, be the fractional power space corresponding to A_ε, $\varepsilon \geq 0$. Then, for $p=\frac{2d}{d-4\alpha}$, we have the following embeddings,*

$$X_\varepsilon^\alpha \hookrightarrow L^p(Q), \ \textit{for } \varepsilon>0, \qquad \textit{and} \qquad X_0^\alpha \hookrightarrow L^p(0,1), \ \textit{for } \varepsilon=0,$$

with embedding constants independent of ε.

Proof. It is known, the operators A_ε, $\varepsilon \geq 0$, see Section 3.4, are selfadjoint. Then, by [2], section 4.7, the purely imaginary powers are bounded, more precisely, they are unitary operators, that is,

$$\|A_\varepsilon^{it}\|_{\mathcal{L}(L^2(Q),L^2(Q)} \leq 1, \qquad t\in\mathbb{R} \qquad \text{and} \quad 0<\varepsilon\leq\varepsilon_0,$$

and,

$$\|A_0^{it}\|_{\mathcal{L}(L_g^2(0,1),L_g^2(0,1))} \leq 1, \qquad t\in\mathbb{R} \qquad \text{and} \quad \varepsilon=0.$$

Then, since purely imaginary powers are bounded, by [55], Theorem 1.15.3, we have the characterization of the fractional power space X_ε^α, with $0<\alpha<\frac{1}{2}$, via complex interpolation space as follows,

$$X_\varepsilon^\alpha = [X_\varepsilon^0, X_\varepsilon^{\frac{1}{2}}]_{2\alpha} = [L^2(Q), H_{\boldsymbol{\varepsilon}}^1(Q)]_{2\alpha}, \qquad 0<2\alpha<1.$$

Moreover, since these imaginary powers are uniformly bounded, (they are unitary), by [2], section 2.9, and [51], Theorem 3, the norm of the fractional power space X_ε^α and the norm of the interpolation space $[L^2(Q), H_{\boldsymbol{\varepsilon}}^1(Q)]_{2\alpha}$ are uniform equivalent in ε. We mean, the constant C involved in this equivalence,

$$C^{-1}\|\cdot\|_{X_\varepsilon^\alpha} \leq \|\cdot\|_{[L^2(Q),H_{\boldsymbol{\varepsilon}}^1(Q)]_{2\alpha}} \leq C\|\cdot\|_{X_\varepsilon^\alpha}, \tag{3.6.1}$$

is uniform in ε.

We also have that $H_{\boldsymbol{\varepsilon}}^1(Q) \hookrightarrow H^1(Q) \hookrightarrow L^{\frac{2d}{d-2}}(Q)$ and, obviously, $L^2(Q) \hookrightarrow L^2(Q)$, both embeddings with constants uniform in ε. So, by interpolation theory we have $[L^2(Q), H_{\boldsymbol{\varepsilon}}^1(Q)]_{2\alpha} \hookrightarrow [L^2(Q), L^{\frac{2d}{d-2}}(Q)]_{2\alpha}$ with constant embedding uniform in ε.

Moreover, whenever $\phi \in [L^2(Q), H^1_{\boldsymbol{\varepsilon}}(Q)]_{2\alpha}$, we have $\phi \in [L^2(Q), H^1(Q)]_{2\alpha}$. By [55], section 4.3.1, we have that

$$H^{2\alpha}(Q) = [L^2(Q), H^1(Q)]_{2\alpha}, \qquad \text{for} \quad 0 < 2\alpha < 1.$$

Since $d \geq 2$ and $\alpha < \frac{1}{2}$, then $2\alpha < \frac{d}{2}$ and hence, the known Sobolev embedding says, see [49],

$$H^{2\alpha}(Q) \hookrightarrow L^p(Q),$$

for $p = \frac{2d}{d-4\alpha}$.

Then, for $p = \frac{2d}{d-4\alpha}$ and α, $0 < 2\alpha < 1$, since (3.6.1) holds in an uniform way, we have,

$$X^\alpha_\varepsilon \hookrightarrow L^p(Q),$$

with embedding constant uniform in ε.

The "uniform embedding",

$$X^\alpha_0 \hookrightarrow L^p(0,1)$$

is obtained with the same arguments.

■

Appendix A

Morse-Smale and Lischitz shadowing

In this appendix we go over the proof that a Morse-Smale gradient like map in $\mathbb{R}^m$ has the Uniform Lipschitz Shadowing property. We refer to Chapter 1 for definitions of both concepts. The proof of this result can be found in [42]. Although the result is known we believe it will be good to have it written down a complete proof of it.

Assume $T \in \mathcal{KC}^r(\mathbb{R}^m, \mathbb{R}^m)$, $r \geq 1$, is a Morse-Smale gradient like map, that is, the non-wandering set, see Definition 1.1.14 consists of a finite number of hyperbolic fixed points. Let U be a subset of $\mathbb{R}^m$. We define

$$O^+(U) = \bigcup_{n\geq 0} T^n(U), \qquad O^-(U) = \bigcup_{n\leq 0} T^n(U),$$

and

$$O(U) = O^+(U) \cup O^-(U).$$

We denote by $\{p_1, p_2, ..., p_N\}$ the hyperbolic fixed points of T, by $\mathbf{S}(p_i)$, $\mathbf{U}(p_i)$ the stable and unstable linear manifolds around p_i, $i = 1, ..., N$, and by $\mathcal{N}(\mathcal{A})$ a neighborhood of the attractor $\mathcal{A}$ of T.

Remark A.0.2. *Recall that, a well known result is that one has the following equalities for a hyperbolic fixed point p, see for example [53]:*

$$T_pW^s(p) = \mathbf{S}(p) \qquad \textit{and} \quad T_pW^u(p) = \mathbf{U}(p),$$

with $T_pW^s(p)$, (resp. $T_pW^u(p)$), the tangent space of the stable, (resp. unstable), manifold of the hyperbolic fixed point p at p.

Then, we can show the following:

Lemma A.0.3. *There exist neighborhoods $\{V_1, ..., V_N\}$ of $\{p_1, ..., p_N\}$ respectively, such that:*

(i) $V_i \cap V_j = \emptyset$ *for* $i \neq j$.

(ii) *If* $z \in V_i$ *then there exists* $n = n(z)$ *such that if* $T^k(z) \in V_i$ *for* $1 \leq k < n(z)$ *and* $T^{n(z)}(z) \notin V_i$ *then for all* $k > n(z)$, $T^k(z) \notin V_i$.

(iii) *If we define*

$$\tau(z) = \#\{k : T^k(z) \notin \bigcup_{i=1}^{N} V_i\},$$

then there exists a finite number $\mathbf{T_0} > 0$, *usually called the Birkhoff constant, such that*

$$\sup_{z \in \mathcal{N}(\mathcal{A})} \tau(z) \leq \mathbf{T_0}.$$

Proof. Item (i) holds taking the neighborhoods $\{V_1, ..., V_N\}$ of $\{p_1, ..., p_N\}$ small enough.

To prove (ii) we argue as follows. Suppose that for some $i \in \{1, ..., N\}$ there exists a sequence $z_k \in \mathcal{N}(\mathcal{A})$ such that

$$|z_k - p_i| < \frac{1}{k},$$

and numbers $n_k^1 < n_k^2$ with

$$T^{n_k^1}(z_k) \notin V_i \qquad \text{and} \qquad \left|T^{n_k^2}(z_k) - p_i\right| < \frac{1}{k}.$$

Then, we can always take an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$T^{n_k^2}(z_k) \in B_\varepsilon(p_i) \qquad \text{and} \qquad T^{n_k^1}(z_k) \notin B_{\varepsilon_0}(p_i).$$

But this contradicts the property (2) of definition of a gradient map, which implies (ii).

In order to prove (iii) we assume there exists a sequence $z_k \in \mathcal{N}(\mathcal{A})$ and a sequence of numbers $n_k > 0$ with $n_k \to \infty$ as $k \to \infty$ such that

$$\{T^n(z_k) : 0 \leq n \leq 2n_k\} \bigcap \cup_{i=1}^{N} V_i = \emptyset.$$

It is known that

$$dist(T^{n_k}(z_k), \mathcal{A}) \to 0 \qquad \text{as} \quad k \to \infty.$$

Then, for each k, we have

$$dist(T^{n_k}(z_k), \mathcal{A}) = dist(T^{n_k}(z_k), v_k),$$

for some $v_k \in \mathcal{A}$. So, since $\mathcal{A}$ is compact, there exists a sequence $\{v_k\} \in \mathcal{A}$ with a convergent subsequence v_{k_j}, that is,

$$\lim_{j \to \infty} v_{k_j} = v_0. \qquad \text{with} \quad v_0 \in \mathcal{A}.$$

So,

$$T^{n_{k_j}}(z_{k_j}) \to v_0 \in \mathcal{A}, \quad \text{as} \quad j \to \infty.$$

This is obtained as follows

$$dist(T^{n_{k_j}}(z_{k_j}), v_0) \leq dist(T^{n_{k_j}}(z_{k_j}), v_{k_j}) + dist(v_{k_j}, v_0),$$

and we know

$$dist(T^{n_{k_j}}(z_{k_j}), \mathcal{A}) = dist(T^{n_{k_j}}(z_{k_j}), v_{k_j}) \to 0 \quad \text{as} \quad j \to \infty,$$

and

$$dist(v_{k_j}, v_0) \to 0 \quad \text{as} \quad j \to \infty,$$

then,

$$\lim_{j \to \infty} T^{n_{k_j}}(z_{k_j}) = v_0 \in \mathcal{A}.$$

Since $T(\mathcal{A}) = \mathcal{A}$, there exists a global orbit of T through v_0 which belongs to $\mathcal{A}$, that is,

$$\{T^n(v_0) : n \in \mathbb{Z}\} \subset \mathcal{A}.$$

For $n \in \mathbb{Z}^+$ with $n \leq n_k$, we have

$$T^{n+n_k}(z_k) \to T^n(v_0) \quad \text{as} \quad k \to \infty$$

and by hypothesis

$$\{T^{n+n_k}(z_k) \quad \text{with} \quad n \leq n_k\} \bigcap \cup_{i=1}^{N} V_i = \emptyset.$$

This implies

$$\{T^n(v_0) : n \leq n_k\} \bigcap \cup_{i=1}^{N} V_i = \emptyset \quad \text{as} \quad k \to \infty,$$

which contradicts the definition of gradient like map, see [15].

■

Before to introduce the main results of this subsection, we see some definitions needed to continue with our work.

Definition A.0.4. *The* **tangent bundle** *of a differential manifold M is the disjoint union of all tangent spaces to M. That is, the set of pairs*

$$\bigsqcup_{x \in M} T_x M = \{(x, v) : x \in M \quad and \quad v \in T_x M\},$$

where $T_x M$ denote de tangent space to M at x.

Remark A.0.5. *For $M = \mathbb{R}^m$ the tangent bundle is $\mathbb{R}^m \times \mathbb{R}^m$.*

With this, we can prove the following structural stability result from [47].

Lemma A.0.6. *Let $\{p_1, p_2, ..., p_N\}$ be the hyperbolic fixed points of T. For each $i \in \{1, ..., N\}$ fixed, there exist a neighborhood V_i of the fixed point p_i, satisfying Lemma A.0.3, continuous subbundle $\{S_i, U_i\}$ of $\mathbb{R}^m \mid_{\overline{V_i} \cup O(V_i)}$, and a number $\lambda_0 \in (0,1)$ such that*

(1) S_i, U_i are $DT-$ invariant, that is,

$$DT(x)S_i(x) = S_i(T(x)), \qquad \textit{for all} \quad x \in \overline{V_i} \cup O(V_i),$$

and

$$DT(x)U_i(x) = U_i(T(x)), \qquad \textit{for all} \quad x \in \overline{V_i} \cup O(V_i),$$

(2)

$$S_i(x) \oplus U_i(x) = \mathbb{R}^m, \qquad \forall x \in V_i,$$

(3) $S_i(p_i) = \mathbf{S}(p_i)$ and $U_i(p_i) = \mathbf{U}(p_i)$, for $i = 1, ..., N$, with $\{\mathbf{S}(p_i), \mathbf{U}(p_i)\}$, the linear stable and unstable manifolds of the fixed point p_i of T, see definition (1.1.9),

(4) $S_i(x) \subset S_j(x)$, $U_i(x) \supset U_j(x)$ for $x \in O^+(V_i) \cap O^-(V_j)$,

(5) for $x \in V_i$, $v^s \in S_i(x)$, and $v^u \in U_i(x)$ we have

$$|DT(x)v^s| \leq \lambda_0 |v^s|,$$

$$|DT^{-1}(x)v^u| \leq \lambda_0 |v^u|.$$

Proof. In [47] there is a sketched proof of this result. For a more detailed proof of it, we refer to [42], Section 2.2.2. Broadly speaking, the proof consists in transferring the hyperbolic structure, which there exists in the neighborhoods of each fixed point, to its orbits. The author constructs the mentioned continuous subbundles using the transversality of the stable and unstable manifolds, techniques of the "λ-lemma" and some compact properties that finite dimension provides.

■

Once we have for each $i \in \{1, ..., N\}$ the splitting $\{S_i, U_i\}$ in the closure of the neighborhood of p_i, $\overline{V}_i$, and in its orbit, $O(V_i)$, our next step is to extend it to a neighborhood of the attractor $\mathcal{A}$. This geometric structure on the attractor $\mathcal{A}$, is the key to obtain the results we present here. To construct this structure we proceed as follows. Let V be

$$V = \bigcup_{i=1}^{N} V_i,$$

where $\{V_i\}_{i=1}^{N}$ are the neighborhoods of the hyperbolic fixed points $\{p_i\}_{i=1}^{N}$ of Lemma A.0.6. We fix a Birkhoff constant $\mathbf{T_0}$ for V. Let $q \in \mathcal{N}(\mathcal{A})$ fixed. Denote by $i(q)$ the integer number $i \in \{1, ..., N\}$, so that $V_{i(q)}$ is the first neighborhood visited by the positive orbit of q. That is, if $q \in V_i$ then $i(q) = i$ or if $q \notin V_j$ for all $j \in \{1, ..., N\}$ then there exists n_0, $n_0 \in \{1, ..., \mathbf{T_0}\}$, such that $y = T^{n_0}(q) \in V_{i(q)}$ for some $i(q) \in \{1, ..., N\}$ and $T^n(q) \notin \cup_{i=1}^{N} V_i$, for $n = 1, 2, ..., n_0 - 1$.

With this, for all $q \in \mathcal{N}(\mathcal{A})$ we define a family of linear subspaces of $\mathbb{R}^m$ by,

$$S(q) = D(T^{-n_0})(y)S_{i(q)}(y)$$

and

$$U(q) = D(T^{-n_0})(y)U_{i(q)}(y).$$

Observe that if $q \in V_i$ then $S(q) = S_i$ and $U(q) = U_i$.

This family satisfies the following properties.

Lemma A.0.7. *Let $q \in \mathcal{N}(\mathcal{A})$. The family of linear subspaces of $\mathbb{R}^m$, $(\{S(q)\}_{q\in\mathcal{N}(\mathcal{A})}$, $\{U(q)\}_{q\in\mathcal{N}(\mathcal{A})})$ described above, satisfies that*

(1)

$$D(T^n)(q)S(q) \subset S(T^n(q)) \qquad \forall n = 0, 1, 2, ...$$

$$D(T^{-n})(q)U(q) \subset U(T^{-n}(q)) \qquad \forall n = 0, 1, 2, ...$$

(2) There exists a constant $C > 0$ such that

$$|D(T^n)(q)v^s| \leq C\lambda_0^n|v^s| \quad with \quad v^s \in S(q), \quad \forall n \geq 0$$

$$|D(T^{-n})(q)v^u| \leq C\lambda_0^n|v^u| \quad with \quad v^u \in U(q), \quad \forall n \geq 0,$$

with $\lambda_0 \in (0, 1)$ of Lemma A.0.6.

(3) There exists $\mathcal{C} > 0$ such that

$$\|P(q)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)}, \|Q(q)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \mathcal{C}, \qquad q \in \mathcal{N}(\mathcal{A}),$$

where $P(q)$, $Q(q)$ are the projectors in $\mathbb{R}^m$ onto $S(q)$ parallel to $U(q)$, and $U(q)$ parallel to $S(q)$, respectively.

(4)

$$S(q) \oplus U(q) = \mathbb{R}^m, \quad for \quad q \in \mathcal{N}(\mathcal{A}).$$

Proof. The proof of (1) is a direct consequence of Lemma A.0.6, points (1) and (3). To prove (2) we start with $n = 0$. For $n = 0$,

$$DT^0(q) = I,$$

so the desired estimate is obvious. We consider $n > 0$ and $\{V_i\}_{i=1}^N$ the neighborhoods of Lemma A.0.6. We suppose the orbit $T^n(q)$ visits the neighborhoods $\{V_1, V_2, ..., V_M\}$, with $M \leq N$, in this order. Then, there exist subintervals $(a_i, b_i) \in [0, n]$, $i = 1, ..., M$ and $M \leq N$, see Figure (A), with a_i defined as follows

$$\{a_i \in [0, n] : T^{a_i}(q) \in V_i \quad \text{and} \quad T^s(q) \notin V_i \quad \text{for} \quad a_i < s \leq n\},$$

and b_i,

$$\{b_i \in [0, n] : T^{b_i}(q) \in V_i \qquad \text{and} \quad T^s(q) \notin V_i \quad \text{for} \quad 0 \leq s < b_i\}.$$

We just prove one case. The case such that $q \in V$ and $y = T^n(q) \in V$. The other cases are shown in a similar way, taking into account the time $T^s(q) \notin V$ is a finite time less than $\mathbf{T_0}$. Then,

$$\sum_{i=1}^{M-1} (b_{i+1} - a_i) \leq M\mathbf{T_0}. \tag{A.0.1}$$

We denote $\mathbf{N_0} = M\mathbf{T_0}$.

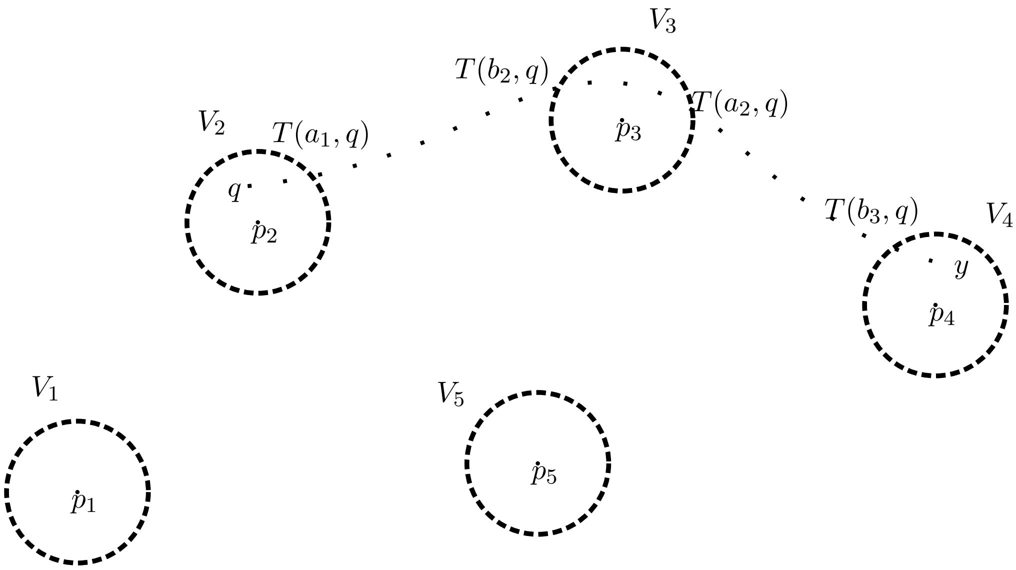

Figure A.1: N=5

Take $v \in S(q)$. We define

$$v_i^- = DT^{a_i}(q)v, \qquad v_i^+ = DT^{b_i}(q)v \qquad \text{and} \quad w = DT^n(q)v.$$

Since for $m \in [0, a_1] \cup [b_2, a_2] \cup [b_3, a_3] \cup ... \cup [b_M, n]$, $T^m(q) \in V$, and so, by Lemma A.0.6 and known estimates,

$$\|D(T^n)(u)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq e^{C_0|n|}, \quad \forall \quad n \in \mathbb{Z} \quad \text{and} \quad u \in \mathcal{A}$$

with C_0 a constant, we have

$$|v_1^-| \leq \lambda_0^{a_1}|v|, \qquad |v_2^-| \leq \lambda_0^{a_2-b_2}|v_2^+|, \quad ... \quad, |w| \leq \lambda_0^{n-b_M}|v_M^+|,$$

and

$$|v_2^+| \leq e^{C_0(b_2-a_1)}|v_1^-|, \quad |v_3^+| \leq e^{C_0(b_3-a_2)}|v_2^-|, \quad ... \quad |v_M^+| \leq e^{C_0(b_M-a_{M-1})}|v_{M-1}^-|.$$

If we put everything together, we obtain

$$|w| \leq \lambda_0^{n-b_M}|v_M^+| \leq ... \leq \lambda_0^{n-b_M+a_{M_1}-...-b_2+a_1} e^{C_0[(b_M-a_{M-1})+...+(b_2-a_1)]}|v|,$$

by (A.0.1),

$$|w| \leq \lambda_0^{n-(\mathbf{N_0})} e^{C_0\mathbf{N_0}}|v|.$$

Taking $C = \lambda_0^{-\mathbf{N_0}} e^{C_0\mathbf{N_0}}$, we conclude

$$|DT^n(q)v| = |w| \leq C\lambda_0^n|v|, \qquad v \in S(q),$$

as we wanted to prove. To obtain the other inequality,

$$|D(T^n(q))v^u| \leq C\lambda_0^{-n}|v^u| \quad \textit{with} \quad v^u \in U(q), \quad \forall n \leq 0,$$

the proof is similar. The difference is that now one has to apply the property (5) of Lemma (A.0.6) related to the subspces U_i for the v_i^- defined as above.

The proof of (4) follows directly from Lemma A.0.6, item (2).

■

A.0.1. Morse-Smale System implies Lipschitz Shadowing

In this section we present some known results from [41, 42]. These results apply the good structure obtained in the previous section to give a relation between Morse-Smale sytems in $\mathbb{R}^m$ and Lipschitz Shadowing property. The goal of this section is to prove that any Morse-Smale map T in $\mathbb{R}^m$ has the Lipschitz Shadowing property, i.e., any negative pseudo-trajectory of T is shadowed by a negative trajectory of T and the shadowing is Lipschitz with respect to the "pseudo coefficient". Moreover, we are interested in proving this Lipschitz Shadowing property is uniform in an appropriate sense in a C^1-neighborhood of the map T.

We start with an abstract shadowing result. Let $\{E_k\}_{k\in\mathbb{Z}}$ be a sequence of Banach spaces and consider a sequence of mappings $\{\phi_k\}_{k\in\mathbb{Z}}$

$$\phi_k : E_k \to E_{k+1}$$

such that,

$$\phi_k(v) = A_k v + \omega_{k+1}(v),$$

with A_k linear mappings and ω_{k+1} probably nonlinear. Then, we have the next result, which can be found in [41, 42].

Lemma A.0.8. *Assume that*

(1) There exist linear projectors

$$P_k, Q_k : E_k \to E_k$$

and numbers $\lambda_1 \in (0,1)$, $\mathcal{C} > 0$ such that

$$\|P_k\|_{\mathcal{L}(E_k,E_k)}, \|Q_k\|_{\mathcal{L}(E_k,E_k)} \leq \mathcal{C}, \quad P_k + Q_k = I$$

and

$$\|A_k\,|_{P_k(E_k)}\,\|_{\mathcal{L}(E_k,E_{k+1})} \leq \lambda_1, \quad A_k P_k(E_k) \subset P_{k+1}(E_{k+1})$$

with I the identity.

(2) There exist linear mappings $B_k : Q_{k+1}(E_{k+1}) \to E_k$ such that

$$B_k Q_{k+1}(E_{k+1}) \subset Q_k(E_k), \quad \|B_k\|_{\mathcal{L}(E_{k+1},E_k)} \leq \lambda_1, \quad A_k B_k\,|_{Q_{k+1}(E_{k+1})}= I.$$

(3) There exist numbers $\mathbf{C}$, $\Delta > 0$ such that

$$\|\omega_{k+1}(v) - \omega_{k+1}(v')\|_{E_{k+1}} \leq \mathbf{C}\|v - v'\|_{E_k} \quad for \quad \|v\|_{E_k}, \|v'\|_{E_k} \leq \Delta.$$

Assume also that the next inequality

$$\mathbf{C}N_1 < 1$$

is satisfied with

$$N_1 = \mathcal{C}\frac{1+\lambda_1}{1-\lambda_1}.$$

Then there exist constants d_0, $L > 0$ and points $v_k \in E_k$ such that, if

$$\|\phi_k(0)\|_{E_{k+1}} \leq d \leq d_0,$$

then

$$\phi_k(v_k) = v_{k+1} \qquad and \qquad \|v_k\|_{E_k} \leq Ld, \qquad k \in \mathbb{Z}.$$

Proof. The proof of this result is obtained by applying the Banach fixed point Theorem. Let Z be the Banach space of sequences $\mathbf{v} = \{v_k\}_{k\geq 0}$ with $v_k \in E_k$ and the norm

$$\|\mathbf{v}\|_\infty = \sup_{k\geq 0} \|v_k\|_{E_k}.$$

We define the operator $\mathcal{H}$ on Z by

$$\mathcal{H}(\mathbf{w}) = \mathbf{v} = \mathbf{v}^1 + \mathbf{v}^2 + \mathbf{v}^3, \qquad \mathbf{w} \in Z,$$

where $\mathbf{v}^i = \{v_k^i\}_{k\geq 0}$,

$$v_k^1 = P_k w_k,$$

$$v_k^2 = \sum_{j=0}^{k-1} A_{k-1}\cdots A_j P_j w_j,$$

$$v_k^3 = -\sum_{j=k}^{\infty} B_k \cdots B_j Q_{j+1} w_{j+1}.$$

First of all, let us show $\mathcal{H}$ maps Z into itself and estimate its norm. To prove this we take $\mathbf{w} \in Z$. From the above definition,

$$\|\mathbf{v}^1\|_\infty \leq \mathcal{C}\|\mathbf{w}\|_\infty.$$

On the other hand, since $P_k w_k \in S_k$ then

$$\|v_n^2\|_{E_n} \leq \mathcal{C}\sum_{k=0}^{n-1} \lambda_1^{n-k}\|w_k\|_{E_k} \leq \mathcal{C}\frac{\lambda_1}{1-\lambda_1}\|\mathbf{w}\|_\infty,$$

and in a similar way we obtain

$$\|v_n^3\|_{E_n} \leq \mathcal{C}\frac{\lambda_1}{1-\lambda_1}\|\mathbf{w}\|_\infty.$$

For this reason

$$\|\mathcal{H}\|_{\mathcal{L}(Z,Z)} \leq \mathcal{C} + \mathcal{C}\frac{\lambda_1}{1-\lambda_1} + \mathcal{C}\frac{\lambda_1}{1-\lambda_1} = \mathcal{C}\frac{1+\lambda_1}{1-\lambda_1} = N_1. \tag{A.0.2}$$

Let $\mathbf{v} = \mathcal{H}(\mathbf{w})$. If we calculate $A\mathbf{v}$ we obtain the following equalities,

$$A_n v_n^1 = A_n P_n w_n$$

$$A_n v_n^2 = \sum_{k=0}^{n-1} A_n \cdots A_k P_k w_k,$$

$$A_n v_n^3 = -[A_n B_n Q_{n+1} w_{n+1} + \sum_{k=n+1}^{\infty} B_{n+1}\cdots B_k Q_{k+1} w_{k+1}] =$$

$$= -(I - P_{n+1})w_{n+1} - \sum_{k=n+1}^{\infty} B_{n+1} \cdots B_k Q_{k+1} w_{k+1}.$$

So,

$$A_n v_n = \underbrace{A_n P_n w_n + \sum_{k=0}^{n-1} A_n \cdots A_k P_k w_k}_{v_{n+1}^2} \underbrace{-(I - P_{n+1})w_{n+1}}_{-w_{n+1}+v_{n+1}^1} \underbrace{- \sum_{k=n+1}^{\infty} B_{n+1} \cdots B_k Q_{k+1} w_{k+1}}_{v_{n+1}^3}.$$

That is,

$$A_n v_n = -w_{n+1} + v_{n+1}^1 + v_{n+1}^2 + v_{n+1}^3 = -w_{n+1} + v_{n+1},$$

hence,

$$v_{n+1} = A_n v_n + w_{n+1}, \qquad n \geq 0.$$

We take

$$L = \frac{N_1}{1 - \mathbf{C}N_1}, \qquad d_0 = \frac{\Delta}{L}. \tag{A.0.3}$$

The equalities,

$$\phi_k(v_k) = v_{k+1}, \qquad k \geq 0,$$

are equivalent to

$$v_{k+1} = A_k v_k + w_{k+1}(v_k) \qquad k \geq 0. \tag{A.0.4}$$

For $\mathbf{v} \in Z$ we define

$$\mathbf{w}(\mathbf{v}) = \{w_k(v_{k-1}), k \geq 0\} \qquad \text{with} \quad w_0 = 0.$$

Then, a solution $\mathbf{v}$ of the equation

$$\mathbf{v} = \mathcal{H}\mathbf{w}(\mathbf{v}),$$

satisfies

$$\phi_k(v_k) = v_{k+1}.$$

Suppose

$$\|\phi_k(0)\|_{E_{k+1}} \leq d.$$

Let $\mathbf{B}$ be the ball $\{\|\mathbf{v}\|_\infty \leq Ld\}$ in Z of radius Ld, remember ($Ld \leq Ld_0 = \Delta$). So for $\mathbf{v}$, $\mathbf{v}' \in \mathbf{B}$, $\|\mathbf{v}\|_\infty, \|\mathbf{v}'\|_\infty \leq \Delta$, and then, by hypothesis (3) of our lemma and (A.0.2), we have

$$\|\mathcal{H}\mathbf{w}(\mathbf{v}) - \mathcal{H}\mathbf{w}(\mathbf{v}')\|_\infty \leq N_1\|\mathbf{w}(\mathbf{v}) - \mathbf{w}(\mathbf{v}')\|_\infty \leq N_1\mathbf{C}\|\mathbf{v} - \mathbf{v}'\|_\infty.$$

From (A.0.4) we obtain

$$\|w_{k+1}(0)\|_{E_{k+1}} = \|v_{k+1}\|_{E_{k+1}} = \|\phi_k(0)\|_{E_{k+1}} \leq d,$$

then, for $\mathbf{v} \in \mathbf{B}$

$$\|\mathcal{H}\mathbf{w}(\mathbf{v})\|_\infty \leq \|\mathcal{H}\mathbf{w}(0)\|_\infty + \|\mathcal{H}\mathbf{w}(\mathbf{v}) - \mathcal{H}\mathbf{w}(0)\|_\infty \leq N_1 d + N_1 \mathbf{C} \|\mathbf{v}\|_\infty \leq N_1 d + N_1 \mathbf{C} L d =$$

$$= d(N_1 + N_1 \mathbf{C} L) = Ld,$$

last equality is because of (A.0.3). We have proved $\mathcal{H}\mathbf{w}$ maps $\mathbf{B}$ into itself and by hypothesis (3) it is a contraction. So, by the Banach fixed point theorem, $\mathcal{H}\mathbf{w}$ has a fixed point, ie., there exist a solution of the equation

$$\mathbf{v} = \mathcal{H}\mathbf{w}(\mathbf{v}),$$

which satisfies

$$\|\mathbf{v}\|_\infty \leq Ld.$$

Remember we have seen that a solution $\mathbf{v}$ of the equation $\mathbf{v} = \mathcal{H}\mathbf{w}(\mathbf{v})$, satisfies $\phi_k(v_k) = v_{k+1}$, then we have proved the desired result.

■

This abstract result join with the linear subspaces of $\mathbb{R}^m$ constructed in Lemma A.0.7 allow us to introduce an important tool of this section.

Lemma A.0.9. *Let T,*

$$T : \mathbb{R}^m \to \mathbb{R}^m,$$

be a Morse-Smale map which has an attractor $\mathcal{A}$. Then T has the Lipschitz Shadowing property on a neighborhood $\mathcal{N}(\mathcal{A})$ of its attractor. That is, there exist constants L and d_0, small enough such that, for all $d \leq d_0$, any negative d-pseudo-trajectory of T, $\{x_k\}_{k \in \mathbb{Z}^-} \subset \mathcal{N}(\mathcal{A})$, is Ld-shadowed by a negative trajectory of T.

Proof. The proof of this result basically consists in applying Lemma A.0.8. So, we are interested in showing that all the hypotheses of this lemma hold.

First, we prove the Lipschitz property for $(d, \mathbf{N})$-pseudo-trajectories, with $\mathbf{N}$ and $d \leq d_0$ chosen bellow. And then, with an easy step we show the Lipschitz property for d-pseudo-trajectories.

Since T is a Morse-Smale map, then $T \in C^1(\mathbb{R}^m, \mathbb{R}^m)$, see Definition 1.1.15. Let $\mathcal{U}$ be a neighborhood of attractor $\mathcal{A}$, large enough, such that there exists a neighborhood $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ with $\mathcal{N}(\mathcal{A}) \subset \overline{\mathcal{N}(\mathcal{A})} \subset \mathcal{U}$. Then, by previous section, since T is a Morse-Smale map, for all $q \in \mathcal{U}$ there exist linear subspaces $\{S(q), U(q)\}$ which satisfy Lemma A.0.7. In particular, we have

$$|D(T^n)(q)v^s| \leq C\lambda_0^n |v^s| \quad \textit{with} \quad v^s \in S(q), \quad \forall n \geq 0$$

$$|D(T^{-n})(q)v^u| \leq C\lambda_0^n |v^u| \quad \textit{with} \quad v^u \in U(q), \quad \forall n \geq 0,$$

with $\lambda_0 \in (0,1)$. We take $\mu \in (0,1)$ and consider a natural number $\mathbf{N}$, large enough, so that

$$C\lambda_0^{\mathbf{N}-1} \leq \mu. \tag{A.0.5}$$

If we see in detail the proof of Lemma A.0.7, we realize $C = \lambda_0^{-\mathbf{N_0}} e^{C_0 \mathbf{N_0}}$. So, we take $\mathbf{N} > \mathbf{N_0} + 1$, large enough. With this, we have for all $q \in \mathcal{U}$,

$$|D(T^n)(q)v^s| \leq \mu |v^s| \quad with \quad v^s \in S(q), \quad \forall n \geq \mathbf{N} - 1, \tag{A.0.6}$$

$$|D(T^{-n})(q)v^u| \leq \mu |v^u| \quad with \quad v^u \in U(q), \quad \forall n \geq \mathbf{N} - 1, \tag{A.0.7}$$

with $\mu \in (0,1)$. Moreover, we take $K \geq \mathcal{C}$ such that

$$\|D(T^n)(q)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq K \quad \text{ for } \quad |n| \leq \mathbf{N} + 1, \tag{A.0.8}$$

where $\mathcal{C}$ is so that $\|P(q)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)}, \|Q(q)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \mathcal{C}$.

Keeping in mind hypotheses of Lemma A.0.8 and for technical reasons, we take $\nu_0 \in (0,1)$ such that $\lambda = (1+\nu_0)\mu < 1$. Then, for this λ and $\mathcal{C}$ from Lemma A.0.7, we consider $N_1 = \mathcal{C}\frac{1+\lambda}{1-\lambda}$ and look for $\kappa > 0$ such that $\kappa N_1 < 1$. Moreover, we take $\nu \in (0, \nu_0)$ so that

$$K(2K+1)\nu < \frac{\kappa}{2}. \tag{A.0.9}$$

Below we denote by d_0 a positive constant that depends only on $\mathcal{C}, K, \nu, \kappa$. Throughout this proof, we consider $(d, \mathbf{N})$-pseudo-trajectories with $\mathbf{N}$ given by (A.0.5) and $d \leq d_0$ where d_0 is the minimal one previously chosen.

To apply Lemma A.0.8 we need a sequence of Banach spaces $\{E_k\}$ and a sequence of mappings $\{\phi_k\}$. Let $\{x_k\}_{k\in\mathbb{Z}}$ be a $(d,\mathbf{N})$-pseudo-trajectory of T. We take

$$E_k = T_{x_k}\mathbb{R}^m,$$

the tangent space of $\mathbb{R}^m$ at the point x_k, that is, the space $\mathbb{R}^m$ centered at the point x_k.

Let $\mathcal{T}$ be the $\mathbf{N}$-iteration of the map T, $\mathcal{T}(q) = T^{\mathbf{N}}(q)$. Then, we define

$$\phi_k : E_k \to E_{k+1}$$

by

$$\phi_k(v) := \mathcal{T}(x_k + v) - x_{k+1}.$$

So,

$$D\phi_k(0) = D\mathcal{T}(x_k)$$

and

$$D\phi_k(0) : E_k \to E_{k+1}.$$

Note that, we can write

$$\phi_k(v) = D\phi_k(0)v + \underbrace{\phi_k(v) - D\phi_k(0)v}_{h_{k+1}(v)} = D\phi_k(0)v + h_{k+1}(v).$$

Since $\{x_k\}_{k\in\mathbb{Z}}$ is a $(d, \mathbf{N})$-pseudo-trajectory of T,

$$h_{k+1}(0) = \phi_k(0) = \mathcal{T}(x_k) - x_{k+1} = T^{\mathbf{N}}(x_k) - x_{k+1} \neq 0,$$

and $Dh_{k+1}(0) = D\phi_k(0) - D\phi_k(0) = 0$. Moreover,

$$|h_{k+1}(v) - h_{k+1}(v')| = |\mathcal{T}(x_k + v) - \mathcal{T}(x_k + v') - D\mathcal{T}(x_k)(v - v')|.$$

Then, since the derivatives of $\mathcal{T}$ are uniformly continuous on $\overline{\mathcal{N}(\mathcal{A})}$, there exists d_0 such that

$$|h_{k+1}(v) - h_{k+1}(v')| \leq \frac{\kappa}{2}|v - v'| \quad \text{for} \quad v, v' \in \mathbb{R}^m \quad \text{with} \quad |v|, |v'| \leq d_0. \quad \text{(A.0.10)}$$

As we have mentioned before, for $x_k \in \mathcal{N}(\mathcal{A}) \subset \mathcal{U}$, there exist a family of linear subspaces of E_k, $\{S(x_k), U(x_k)\}$ and its corresponding projectors $P(x_k)$, $Q(x_k)$,

$$S(x_k) = P(x_k)E_k,$$

$$U(x_k) = Q(x_k)E_k$$

which satisfies the properties described in Lemma A.0.7.

Moreover, by Lemma A.0.7, we can relate $S(x_k)$ with $S(\mathcal{T}(x_k))$ and $U(x_k)$ with $U(\mathcal{T}(x_k))$. But, to obtain the Lipschitz Shadowing property for T, we need to be able to relate $S(x_k)$ with $S(x_{k+1})$ and $U(x_k)$ with $U(x_{k+1})$. With this purpose, we present the following lemma, a consequence of the second part of Lemma 2.1 in [41]. This result allows us to pass from the subspace $S(\mathcal{T}(x_k))$ to $S(x_{k+1})$ with the application $\mathbf{F}$ and from $U(\mathcal{T}^{-1}(x_{k+1}))$ to $U(x_k)$ with $\mathbf{G}$.

Lemma A.0.10. *For all $\nu > 0$ there exist $d_0 > 0$ such that if $p, y \in \mathbb{R}^m$, $n_1, n_2 \in \mathbb{Z}$, $q = T^{n_1}(p)$, $z = T^{n_2}(y)$ and*

$$dist(y, q) < d_0 \qquad dist(z, p) < d_0$$

then there exist a linear isomorphism $\mathbf{F}(p, y) : T_y\mathbb{R}^m \to T_y\mathbb{R}^m$ with

$$\|\mathbf{F} - I\|_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)} \leq \nu, \qquad \mathbf{F}(p, y)(DT^{n_1}(p)S(p)) \subset S(y)$$

and a linear isomorphism $\mathbf{G}(p, y) : T_p\mathbb{R}^m \to T_p\mathbb{R}^m$ with

$$\|\mathbf{G} - I\|_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)} \leq \nu, \qquad \mathbf{G}(p, y)(DT^{n_2}(y)U(y)) \subset U(p).$$

Now, applying Lemma A.0.10, we fix d_0 such that, if

$$dist(x_k, \mathcal{T}^{-1}(x_{k+1})) < d_0, \qquad dist(\mathcal{T}(x_k), x_{k+1}) < d_0,$$

then, for the linear isomorphisms $\mathbf{F}(x_k, x_{k+1})$ and $\mathbf{G}(x_k, x_{k+1})$, we have

$$\|\mathbf{F} - I\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)}, \|\mathbf{G} - I\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)}, \|\mathbf{G}^{-1} - I\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \nu, \tag{A.0.11}$$

with $\nu \in (0, \nu_0)$, see (A.0.9).

We define

$$A_k^s := \mathbf{F}(x_k, x_{k+1}) D\mathcal{T}(x_k) P(x_k),$$
$$A_k^u := D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))\mathbf{G}^{-1}(x_k, x_{k+1}) Q(x_k),$$

and

$$B_k := \mathbf{G}(x_k, x_{k+1}) D\mathcal{T}^{-1}(x_{k+1}).$$

That is, A_k^s relates the subspace $S(x_k)$ to $S(x_{k+1})$, A_k^u relates $U(x_k)$ to $U(x_{k+1})$ and finally, B_k relates the subspace $U(x_{k+1})$ to $U(x_k)$.

Now $\phi_k(v)$ can be written as follows,

$$\phi_k(v) = A_k v + \omega_{k+1}(v),$$

with

$$A_k = A_k^s + A_k^u,$$

and

$$\omega_{k+1}(v) = \phi_k(v) - A_k v = [D\mathcal{T}(x_k) - A_k]v + h_{k+1}(v).$$

Our next step is to prove that these A_k, B_k and ω_{k+1} satisfy the conditions required in Lemma A.0.8. With this purpose, we take $v^s \in S(x_k) = P(x_k)E_k$. By Lemma A.0.7 and (A.0.6), we have

$$D\mathcal{T}(x_k)v^s \in S(\mathcal{T}(x_k)) \qquad \text{and} \quad |D\mathcal{T}(x_k)v^s| \leq \mu |v^s|.$$

Since (A.0.11) holds and $F(x_k, x_{k+1})(D\mathcal{T}(x_k)v^s) \in S(x_{k+1})$, we have

$$A_k^s S(x_k) \subset S(x_{k+1}),$$

and

$$|F(x_k, x_{k+1})(D\mathcal{T}(x_k)v^s)| \leq (1+\nu)\mu |v^s| \overset{\nu\in(0,\nu_0)}{\leq} \lambda |v^s|$$

with $\lambda = (1 + \nu_0)\mu < 1$ chosen before. Hence,

$$\|A_k^s\,|_{S(x_k)}\,\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \lambda.$$

Now we take $v^u \in U(x_{k+1})$. By Lemma A.0.7 and Lemma A.0.10

$$B_k v^u = \mathbf{G}(x_k, x_{k+1}) D\mathcal{T}^{-1}(x_{k+1}) v^u \subset U(x_k), \tag{A.0.12}$$

and again from (A.0.11) and (A.0.7)

$$|\mathbf{G}(x_k, x_{k+1}) D\mathcal{T}^{-1}(x_{k+1}) v^u| \leq (1+\nu)\mu |v^u| \leq \lambda |v^u|.$$

We have obtained

$$B_k U(x_{k+1}) \subset U(x_k), \qquad \|B_k\,|_{U(x_k)}\,\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \lambda.$$

Let $v^1 = B_k v^u \in U(x_k)$, so $Q(x_k)v^1 = v^1$. Then,

$$A_k^u B_k v^u = D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))\mathbf{G}^{-1}(x_k, x_{k+1}) Q(x_k) B_k \underbrace{B_k^{-1} v^1}_{v^u} =$$

$$D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))\mathbf{G}^{-1}(x_k, x_{k+1}) v^1 = v^u,$$

last equality follows directly from (A.0.12), hence

$$A_k^u B_k\,|_{U(x_{k+1})} = I.$$

Since

$$A_k\,|_{S(x_k)} = A_k^s, \qquad A_k B_k\,|_{U(x_{k+1})} = A_k^u B_k\,|_{U(x_{k+1})},$$

A_k, B_k satisfy the conditions of Lemma A.0.8.

Since $D\mathcal{T}$ is uniformly continuous on $\overline{\mathcal{N}(\mathcal{A})}$, there exists $d_0 > 0$ such that, if

$$dist(x_k, \mathcal{T}^{-1}(x_{k+1})) \leq d_0,$$

then,

$$\|D\mathcal{T}(x_k) - D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \nu. \tag{A.0.13}$$

It remains to be shown ω_{k+1} satisfies the related hypotheses of Lemma A.0.8, that is, if ω_{k+1} holds the Lipschitz condition. Remember

$$\omega_{k+1}(v) = (D\mathcal{T}(x_k) - A_k)v + h_{k+1}(v).$$

Then, for $d \leq d_0$ we first estimate $\|D\mathcal{T}(x_k) - A_k\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)}$,

$$\|D\mathcal{T}(x_k) - A_k\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} = \|D\mathcal{T}(x_k)(P(x_k) + Q(x_k)) - A_k\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq$$

$$\leq \|D\mathcal{T}(x_k)P(x_k) - A_k^s\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} + \|D\mathcal{T}(x_k)Q(x_k) - A_k^u\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq$$

$$\leq \|D\mathcal{T}(x_k)P(x_k) - \underbrace{F(x_k, x_{k+1})D\mathcal{T}(x_k)P(x_k)}_{A_k^s}\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} +$$

$$+\|D\mathcal{T}(x_k)Q(x_k) - \underbrace{D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))G^{-1}(x_k, x_{k+1})Q(x_k)}_{A_k^u}\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)}.$$

From (A.0.8) and (A.0.11), the first term is estimated as follows,

$$\|D\mathcal{T}(x_k)P(x_k) - F(x_k, x_{k+1})D\mathcal{T}(x_k)P(x_k)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq K^2\nu.$$

To estimate the second term we take the chosen $d_0 > 0$ such that (A.0.13) holds. Then,

$$\|D\mathcal{T}(x_k)Q(x_k) - D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))G^{-1}(x_k, x_{k+1})Q(x_k)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq$$

$$K\|D\mathcal{T}(x_k) - D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))G^{-1}(x_k, x_{k+1})\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} =$$

$$K\|D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1}))(G^{-1}(x_k, x_{k+1}) - I) + D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1})) - D\mathcal{T}(x_k)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq$$

$$K^2\nu + K\|D\mathcal{T}(\mathcal{T}^{-1}(x_{k+1})) - D\mathcal{T}(x_k)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq$$

$$\leq K^2\nu + K\nu = \nu K(K+1),$$

we have applied (A.0.13).

If we put all together we obtain

$$\|D\mathcal{T}(x_k) - A_k\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq K^2\nu + \nu K(K+1) = \nu K(2K+1) \leq \frac{\kappa}{2},$$

the last inequality follows from (A.0.9). Since (A.0.10) holds, then we have the desired estimate

$$|\omega_{k+1}(v) - \omega_{k+1}(v')| \leq \kappa|v - v'|.$$

So, for $d \leq d_0$, ϕ_k satisfies the conditions of Lemma A.0.8 with $\Delta = d_0$.

Now we can apply Lemma A.0.8. Let $\mathcal{N}(\mathcal{A})$ be the mentioned neighborhood of the attractor $\mathcal{A}$. Consider $\{x_k\}_{k\in\mathbb{Z}^-} \subset \mathcal{N}(\mathcal{A})$ a negative $(d, \mathbf{N})$-pseudo-trajectory of T, $d \leq d_0$, then, for $k \leq 0$ we have

$$|\phi_k(0)| = |\mathcal{T}(x_k) - x_{k+1}| = |T^{\mathbf{N}}(x_k) - x_{k+1}| \leq d \leq d_0.$$

So, by Lemma A.0.8 there exist points $v_k \in \mathbb{R}^m$, $k \in \mathbb{Z}^-$, such that

$$\phi_k(v_k) = v_{k+1} \tag{A.0.14}$$

and

$$|v_k| \leq Ld. \tag{A.0.15}$$

Let $\mathbf{p_k} = x_k + v_k$. Definition of ϕ_k,

$$\phi_k(v) := \mathcal{T}(x_k + v) - x_{k+1},$$

and (A.0.14) imply

$$\mathcal{T}(\mathbf{p_k}) = \mathbf{p_{k+1}},$$

that is, $\mathbf{p_{k+1}}$ belongs to the trajectory of the point $\mathbf{p_k}$ under T. Hence these points belong to a negative trajectory of T. Moreover, for $k \in \mathbb{Z}^-$,

$$|\mathbf{p_k} - x_k| = |(x_k + v_k) - x_k| = |v_k| \leq Ld, \qquad \text{with} \quad d \leq d_0.$$

So, we have proved T has the Lipschitz Shadowing property for $(d, \mathbf{N})$-pseudo-trajectories on $\mathcal{N}(\mathcal{A})$. To obtain this result for d-pseudo-trajectories, (that is, $(d,1)$-pseudo-trajectories), we argue as follows. Let L_T be the lowest Lipschitz constant for T. Denote $\{y_{k\mathbf{N}}\}_{k\in\mathbb{Z}^-} = \{x_k\}_{k\in\mathbb{Z}^-}$ and $p_{k\mathbf{N}} = T^{k\mathbf{N}}(p)$ with $p = \mathbf{p_0}$ to simplify, that is, $p_{k\mathbf{N}} = T^{k\mathbf{N}}(p) = \mathbf{p_k}$.

Then, for $n \in [k\mathbf{N}, (k+1)\mathbf{N}]$ and since $\{y_{k\mathbf{N}}\}_{k\in\mathbb{Z}^-} = \{x_k\}_{k\in\mathbb{Z}^-}$ is a $(d, \mathbf{N})$-pseudo-trajectory of T, see definition 1.1.2,

$$|T^n(p) - y_n| \leq |T^{n-k\mathbf{N}}(p_{k\mathbf{N}}) - T^{n-k\mathbf{N}}(y_{k\mathbf{N}})| + |T^{n-k\mathbf{N}}(y_{k\mathbf{N}}) - y_n| \leq$$

$$\leq L_T^{\mathbf{N}} L d + d = d(L_T^{\mathbf{N}} L + 1).$$

So, we have found a negative trajectory of T, $\{p_k\}_{k\in\mathbb{Z}^-}$, which $L'd$-shadows the negative d-pseudo-trajectory $\{y_k\}_{k\in\mathbb{Z}^-}$, with $L' = L_T^{\mathbf{N}} L + 1$ and $d \leq d_0$. With this, T has the Lipstichz Shadowing property on $\mathcal{N}(\mathcal{A})$ of parameters L' and d_0. Lemma A.0.9 is proved.

■

Next, we study an uniform result for the Lipschitz Shadowing property. More precisely, we prove that if T is a Morse-Smale map then there exists a neighborhood in the C^1 topology, such that any map in this neighborhood has also the Lipschitz Shadowing property with the same parameters. This result is obtained applying Lemma A.0.8.

Lemma A.0.11. *Let $T : \mathbb{R}^m \to \mathbb{R}^m$ be a Morse-Smale map. There exist a neighborhood Θ of T in the C^1 topology and numbers L, d_0 such that, for any map $T' \in \Theta$, T' has the Lipschitz Shadowing property on a neighborhood of the attractor $\mathcal{A}$ of T, $\mathcal{N}(\mathcal{A})$, with constants L, d_0.*

Proof. To prove this lemma we are going to show that the abstract shadowing result, Lemma A.0.8, holds uniformly, that is, with d_0 and L not depending on $T' \in \Theta$.

With this objective in mind, we proceed as follows. Since T is a Morse-Smale map, as we have proved before, there exists a family of subspaces $\{S(q), U(q)\}_{q\in\mathcal{U}}$, with $\mathcal{U}$ a neighborhood of the attractor of T, $\mathcal{A}$, which satisfies Lemma A.0.7 and the following. Moreover, given $\nu > 0$, $\nu \in (0, \nu_0)$, and $\mathbf{N} > 0$, $\mathbf{N} \in \mathbb{N}$, there exists a $d_0 > 0$ such that, if $p, y \in \mathcal{U}$, $q = T^{\mathbf{N}}(p)$, $z = T^{-\mathbf{N}}(y)$ and

$$d(y, q) < d_0,$$

then there exist linear isomorphisms $\mathbf{F}$ and $\mathbf{G}$, see Lemma A.0.10, such that,

$$\mathbf{F}(p,y) : T_y\mathbb{R}^m \to T_y\mathbb{R}^m, \qquad \|\mathbf{F} - Id\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \nu, \quad \mathbf{F}(p,y)[DT^{\mathbf{N}}(p)S(p)] \subset S(y),$$

and

$$\mathbf{G}(p,y) : T_p\mathbb{R}^m \to T_p\mathbb{R}^m, \qquad \|\mathbf{G}-Id\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \nu, \quad \mathbf{G}(p,y)[DT^{-\mathbf{N}}(y)U(y)] \subset U(p).$$

Then, we fix this family, $\{S(q),U(q)\}_{q\in\mathcal{U}}$, and take $\mathbf{N}\in\mathbb{N}$ large enough such that $C\lambda_0^{\mathbf{N}} < 1$, with C and λ_0 from property (2) of Lemma A.0.7, related to the family $\{S(q),U(q)\}_{q\in\mathcal{U}}$. Furthermore, we fix $d_0 > 0$ which satisfies all the conditions imposed in the proof of Lemma A.0.9. Obviously, we consider the minimal $d_0 > 0$ previously chosen.

First, there exists a neighborhood Θ of T in the C^1 topology such that, every $T' \in \Theta$ has the same Lipschitz constant L_T. Then, we take $T' \in \Theta$ and a $(d,\mathbf{N})$-pseudo-trajectory of it, $\{x_k\}_{k\in\mathbb{Z}} \in \mathcal{U}$, with $d \leq d_0$, that is,

$$|x_{k+n} - T'^n(x_k)| \leq d \quad \text{for} \quad |n| \leq \mathbf{N} \quad \text{with} \quad n\in\mathbb{Z}.$$

Let $\mathcal{T}'$ be the $\mathbf{N}$-iteration of the map T', that is, $\mathcal{T}'(q) = T'^{\mathbf{N}}(q)$ and consider $E_k = T_{x_k}\mathbb{R}^m$. We define $\phi_k : E_k \to E_{k+1}$ in a similar way as in the previous lemma,

$$\phi_k(v) := \mathcal{T}'(x_k+v) - x_{k+1}.$$

Again, we can write

$$\phi_k(v) = D\phi_k(0)v + \phi_k(v) - D\phi_k(0)v = D\phi_k(0)v + h_{k+1}, \tag{A.0.16}$$

with $h_{k+1}(v) = \phi_k(v) - D\phi_k(0)v$.

Hence, we have

$$h_{k+1}(v)-h_{k+1}(v') = \mathcal{T}'(x_k+v)-x_{k+1}-D\phi_k(0)v-(\mathcal{T}'(x_k+v')-x_{k+1}-D\phi_k(0)v') =$$

$$= \mathcal{T}'(x_k+v) - \mathcal{T}'(x_k+v') - D\mathcal{T}'(x_k)(v-v'),$$

note that $D\phi_k(0) = D\mathcal{T}'(x_k)$.

Let $\mathcal{T}$ be the $\mathbf{N}$-iteration of the map T, $\mathcal{T}(q) = T^{\mathbf{N}}(q)$. Since $T\in C^1$ and its derivatives are uniformly continuous on $\overline{\mathcal{N}(\mathcal{A})} \subset \mathcal{U}$, we know that there exists a $\Delta \leq d_0$ such that

$$|\mathcal{T}(x_k+v) - \mathcal{T}(x_k+v') - D\mathcal{T}(x_k)(v-v')| \leq \frac{\kappa}{8}|v-v'| \quad \text{for} \quad |v|,|v'| \leq \Delta,$$

with κ from the proof of Lemma A.0.9. Since Θ is a C^1 neighborhood of T, there exists a neighborhood $\Theta^0 \subset \Theta$ such that for any $T' \in \Theta^0$ we have

$$\|D\mathcal{T}'(x_k) - D\mathcal{T}(x_k)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} \leq \frac{\kappa}{4},$$

and, on $\overline{\mathcal{N}(\mathcal{A})} \subset \mathcal{U}$,

$$|\mathcal{T}'(x_k+v) - \mathcal{T}'(x_k+v') - D\mathcal{T}'(x_k)(v-v')| \leq \frac{\kappa}{4}|v-v'|,$$

for $|v|, |v'| \leq \Delta$ and κ from the proof of Lemma A.0.9.

In order to prove we have the abstract shadowing result, Lemma A.0.8, with constants independent on map $T' \in \Theta^0$, we write ϕ_k as follows,

$$\phi_k(v) = A_k v + \omega_{k+1}(v),$$

with A_k the one considered in the proof of Lemma A.0.9 for $\mathcal{T}$, x_k and x_{k+1} and ω_{k+1} probably nonlinear mappings, such that the hypothesis (3) of this lemma holds. Then, since $D\phi_k(0) = D\mathcal{T}'(x_k)$, applying (A.0.16),

$$\omega_{k+1}(v) = \phi_k(v) - A_k v = D\mathcal{T}'(x_k)v + h_{k+1}(v) - A_k v =$$

$$= D\mathcal{T}(x_k)v + D\mathcal{T}'(x_k)v - D\mathcal{T}(x_k)v + h_{k+1}(v) - A_k v =$$

$$= [D\mathcal{T}(x_k) - A_k]v + [D\mathcal{T}'(x_k) - D\mathcal{T}(x_k)]v + h_{k+1}(v).$$

So,

$$|\omega_{k+1}(v) - \omega_{k+1}(v')| =$$

$$= |[D\mathcal{T}(x_k) - A_k](v - v') + [D\mathcal{T}'(x_k) - D\mathcal{T}(x_k)](v - v') + h_{k+1}(v) - h_{k+1}(v')| \leq$$

$$\leq \frac{\kappa}{2}|v - v'| + \frac{\kappa}{4}|v - v'| + \frac{\kappa}{4}|v - v'| = \kappa|v - v'|.$$

The upper bound

$$|[D\mathcal{T}(x_k) - A_k](v - v')| \leq \frac{\kappa}{2}|v - v'|,$$

was obtained in the proof of Lemma A.0.9.

With this, we have all the hypothesis needed to apply Lemma A.0.8. So, for every $T' \in \Theta^0$, since

$$|\phi_k(0)| = |\mathcal{T}'(x_k) - x_{k+1}| = |T'^{\mathbf{N}}(x_k) - x_{k+1}| \leq d \leq d_0,$$

then, there exist $v_k \in \mathbb{R}^m$ and numbers $L, d_0 > 0$, such that

$$\phi_k(v_k) = v_{k+1} \quad \text{and} \quad |v_k| \leq Ld, \quad \text{with} \quad d \leq d_0.$$

Note that the constants L and d_0 only depend on the bound for the norm of A_k and the constant κ, which are the same that those for the map T, that is, they are independent of the map $T' \in \Theta^0$.

We set $\mathbf{p_k} = x_k + v_k$. Since,

$$\phi_k(v_k) = \mathcal{T}'(x_k + v_k) - x_{k+1} = v_{k+1},$$

we have

$$\mathcal{T}'(\mathbf{p_k}) = \mathbf{p_{k+1}},$$

that is, $\mathbf{p_{k+1}}$ belongs to the trajectory of the point $\mathbf{p_k}$ under T', and

$$|\mathbf{p_k} - x_k| = |x_k + v_k - x_k| = |v_k| \leq Ld, \quad \text{with} \quad d \leq d_0.$$

So, we have proved that for any $T' \in \Theta^0$, T' has Lipschitz Shadowing on $\mathcal{N}(\mathcal{A})$ with constants L and d_0 for $(d, \mathbf{N})$-pseudo-trajectories. Then it remains to prove Lipschitz Shadowing for d-pseudo-trajectories, $d \leq d_0$. We argue as in the above result. We know that for any $T' \in \Theta^0 \subset \Theta$, T' has the same Lipschitz constant L_T. We denote $\{y_{k\mathbf{N}}\}_{k\in\mathbb{Z}^-} = \{x_k\}_{k\in\mathbb{Z}^-}$ and $\{p_{k\mathbf{N}}\}_{k\in\mathbb{Z}^-} = \{T'^{k\mathbf{N}}(\mathbf{p_0})\}_{k\in\mathbb{Z}^-}$. Then, as above, for $n \in [k\mathbf{N}, (k+1)\mathbf{N}]$, $n \in \mathbb{Z}$,

$$|T'^n(\mathbf{p_0}) - y_n| \leq |T'^{n-k\mathbf{N}}(p_{k\mathbf{N}}) - T'^{n-k\mathbf{N}}(y_{k\mathbf{N}})| + |T'^{n-k\mathbf{N}}(y_{k\mathbf{N}}) - y_n| \leq$$

$$\leq L_T^{\mathbf{N}} L d + d = d(L_T^{\mathbf{N}} L + 1).$$

So, for any $T' \in \Theta^0$ we have found a negative trajectory $\{p_n\}_{n\in\mathbb{Z}^-}$ which $L'd$-shadows the negative d-pseudotrajectory, $d \leq d_0$, with $L' = L_T^{\mathbf{N}} L + 1$. Hence, there exists a neighborhood Θ^0 of the map T in the topology C^1 and numbers L', d_0, such that any $T' \in \Theta^0$ has the Lipschitz Shadowing property on $\mathcal{N}(\mathcal{A})$ with the same parameters L' and d_0, independent on T'.

■

Appendix B

Morse-Smale systems and Nonautonomous Inverse Shadowing

In this appendix, we show that a Morse-Smale gradient like map in $\mathbb{R}^m$ has the Non-autonomous Inverse Shadowing property, see Definition 1.2.8. Although properly speaking there is not such a result in the literature, the proof of this result follows very much the steps of the lower continuity part of the proof of Theorem 1 of [35]. We found it convenient to present a proof of this fact.

Before proving this result, we need to present some important lemmas essential to prove Proposition 1.2.14. Throughout this section we consider $T : \mathbb{R}^m \to \mathbb{R}^m$ a Morse-Smale map with its linear subspaces $\{S(q), U(q)\}$ which satisfy Lemma A.0.7.

Lemma B.0.12. *There exist non-negative real numbers $\lambda_s, \lambda_u, \mu, \delta$ with*

$$\lambda_s < 1 < \lambda_u \quad \textit{and} \quad (1-\lambda_s)(\lambda_u - 1) > \mu^2,$$

such that

$$\left|P_{T(x)}(T(x+u+v) - T(x+\tilde{u}+v))\right| \leq \lambda_s |u-\tilde{u}|, \tag{B.0.1}$$

$$\left|P_{T(x)}(T(x+u+v) - T(x+u+\tilde{v}))\right| \leq \mu |v-\tilde{v}|, \tag{B.0.2}$$

$$\left|Q_{T(x)}(T(x+u+v) - T(x+\tilde{u}+v))\right| \leq \mu |u-\tilde{u}|, \tag{B.0.3}$$

$$\left|Q_{T(x)}(T(x+u+v) - T(x+u+\tilde{v}))\right| \geq \lambda_u |v-\tilde{v}| \tag{B.0.4}$$

for all $x \in K$, K a compact subset of $\mathbb{R}^m$, any $u, \tilde{u} \in S(x)$, any $v, \tilde{v} \in \mathbf{U}(x) = DT^{-1}(U(q))$ in estimates (B.0.2) and (B.0.4) and any $v, \tilde{v} \in U(x)$ in (B.0.1) and (B.0.3) such that $|u|, |\tilde{u}|, |v|, |\tilde{v}| \leq \delta$. We have denote $P(T(x))$ and $Q(T(x))$ by $P_{T(x)}$ and $Q_{T(x)}$, respectively, to clarify notation.

Proof. Let r be a point of a compact subset K of $\mathbb{R}^m$. Since the derivative of T, DT, is continuous, (see definition 1.1.15), for all $\varepsilon > 0$ there exists a neighborhood V of r such that

$$\|DT(r) - DT(v)\|_{\mathcal{L}(\mathbb{R}^m,\mathbb{R}^m)} < \varepsilon \quad \text{for} \quad v \in V.$$

So, we have

$$|T(r+u) - T(r+v) - DT(r)(u-v)| \leq \varepsilon |u-v|,$$

for any r, u, v such that $u, v \in V$.

Since K is compact, the derivative DT is uniformly continuous on K. Therefore, for any $\varepsilon > 0$ there exist a $\eta > 0$ such that if $r \in K$ and $|u|, |v| \leq \eta$ then

$$|T(r+u) - T(r+v) - DT(r)(u-v)| \leq \varepsilon |u-v|. \tag{B.0.5}$$

Now we fix μ, λ_s and λ_u satisfying the properties of our lemma and the additional ones

$$\mu + \lambda_0 < \lambda_s \qquad \text{and} \qquad \frac{\mathcal{C}}{\lambda_0} - \mu > \lambda_u, \tag{B.0.6}$$

with λ_0, $\mathcal{C}$ from Lemma A.0.7.

We take $\eta_0 > 0$ such that if $r \in K$ and $|u|, |v| \leq \eta_0$ then (B.0.5) holds with $\varepsilon = \frac{\mu}{\mathcal{C}}$. Let $\eta = \frac{\eta_0}{2}$. Then, this η satisfies the desired properties of our lemma. Let us see.

Take u, $\tilde{u} \in S(r)$ and $v \in U(r)$ such that $|u|, |\tilde{u}|, |v| \leq \eta$. So, we have $|u+v|, |\tilde{u}+v| \leq \eta_0$ and from (B.0.5) we obtain

$$|T(r+u+v) - T(r+\tilde{u}+v) - DT(r)(u-\tilde{u})| \leq \frac{\mu}{\mathcal{C}} |u-\tilde{u}|. \tag{B.0.7}$$

Since $u, \tilde{u} \in S(r)$, $u - \tilde{u} \in S(r)$ and from (1) of Lemma A.0.7, $DT(r)(u-\tilde{u}) \in S(T(r))$. Thus,

$$P_{T(r)}[DT(r)(u-\tilde{u})] = DT(r)(u-\tilde{u})$$

and from (2) of Lemma A.0.7, taking $C = 1$, we conclude

$$|P_{T(r)}[DT(r)(u-\tilde{u})]| \leq \lambda_0 |u-\tilde{u}| \tag{B.0.8}$$

If we put together (B.0.7) and (B.0.8) then

$$\left|P_{T(r)}[T(r+u+v) - T(r+\tilde{u}+v) - DT(r)(u-\tilde{u})]\right| \geq$$

$$\geq \left|P_{T(r)}[T(r+u+v) - T(r+\tilde{u}+v)]\right| - \left|P_{T(r)}[DT(r)(u-\tilde{u})]\right| \geq$$
$$\geq \left|P_{T(r)}[T(r+u+v) - T(r+\tilde{u}+v)]\right| - \lambda_0 |u-\tilde{u}|.$$

Therefore,

$$\left|P_{T(r)}[T(r+u+v) - T(r+\tilde{u}+v)]\right| \leq$$

$$\leq \left| P_{T(r)}[T(r+u+v) - T(r+\tilde{u}+v) - DT(r)(u-\tilde{u})] \right| + \lambda_0 |u-\tilde{u}| \leq$$
$$\leq \mathcal{C}|T(r+u+v) - T(r+\tilde{u}+v) - DT(r)(u-\tilde{u})| + \lambda_0|u-\tilde{u}| \leq$$
$$\overset{(B.0.7)}{\leq} \mathcal{C}\frac{\mu}{\mathcal{C}}|u-\tilde{u}| + \lambda_0|u-\tilde{u}| \overset{(B.0.6)}{\leq} \lambda_s|u-\tilde{u}|,$$

as we wanted to prove.

We follow with the same arguments to get the second inequality. Take $u \in S(r)$ and $v, \tilde{v} \in \mathbf{U}(r)$. As above, from (B.0.5) we have

$$|T(r+u+v) - T(r+u+\tilde{v}) - DT(r)(v-\tilde{v})| \leq \frac{\mu}{\mathcal{C}}|v-\tilde{v}|. \tag{B.0.9}$$

Since $v, \tilde{v} \in \mathbf{U}(r)$, $v - \tilde{v} \in \mathbf{U}(r)$ then $DT(r)(v-\tilde{v}) \in U(T(r))$ and $P_{T(r)}[DT(r)(v-\tilde{v})] = 0$.

So,

$$\left| P_{T(r)}[T(r+u+v) - T(r+u+\tilde{v}) - DT(r)(v-\tilde{v})] \right| = \left| P_{T(r)}[T(r+u+v) - T(r+u+\tilde{v})] \right|,$$

it follows directly that

$$\left| P_{T(r)}[T(r+u+v) - T(r+u+\tilde{v})] \right| = \left| P_{T(r)}[T(r+u+v) - T(r+u+\tilde{v}) - DT(r)(v-\tilde{v})] \right|$$

$$\leq \mathcal{C}|T(r+u+v) - T(r+u+\tilde{v}) - DT(r)(v-\tilde{v})| \overset{(B.0.9)}{\leq} \mathcal{C}\frac{\mu}{\mathcal{C}}|v-\tilde{v}| = \mu|v-\tilde{v}|,$$

the desired result.

To obtain (B.0.3), we take again $u, \tilde{u} \in S(r)$ and $v \in U(r)$. Then we have

$$|T(r+u+v) - T(r+\tilde{u}+v) - DT(r)(u-\tilde{u})| \leq \frac{\mu}{\mathcal{C}}|u-\tilde{u}|, \tag{B.0.10}$$

and $DT(r)(u-\tilde{u}) \in S(T(r))$ so,

$$Q_{T(r)}[DT(r)(u-\tilde{u})] = 0.$$

Therefore, using the same tools as before we obtain

$$\left| Q_{T(r)}[T(r+u+v) - T(r+\tilde{u}+v)] \right| \leq \mathcal{C}\frac{\mu}{\mathcal{C}}|u-\tilde{u}| = \mu|u-\tilde{u}|.$$

To finish the proof of Lemma B.0.12, take $u \in S(r)$ and $v, \tilde{v} \in \mathbf{U}(r)$. We have

$$|T(r+u+v) - T(r+u+\tilde{v}) - DT(r)(v-\tilde{v})| \leq \frac{\mu}{\mathcal{C}}|v-\tilde{v}|. \tag{B.0.11}$$

And also we have $DT(r)(v-\tilde{v}) \in U(T(r))$, so

$$Q_{T(r)}[DT(r)(v-\tilde{v})] = DT(r)(v-\tilde{v}),$$

from (2) of Lemma A.0.7,

$$\left|Q_{T(r)}[DT(r)(v-\tilde{v})]\right| = |DT(r)(v-\tilde{v})| \geq \frac{1}{\lambda_0}|v-\tilde{v}|. \tag{B.0.12}$$

Thus,

$$\left|Q_{T(r)}[T(r+u+v)-T(r+u+\tilde{v})-DT(r)(v-\tilde{v})]\right| \geq$$

$$\geq \left|Q_{T(r)}[T(r+u+v)-T(r+u+\tilde{v})]\right| - \left|Q_{T(r)}[DT(r)(v-\tilde{v})]\right|.$$

Applying (B.0.12) and (B.0.11),

$$\left|Q_{T(r)}[T(r+u+v)-T(r+u+\tilde{v})]\right| \geq \mathcal{C}(\frac{1}{\lambda}-\frac{\mu}{\mathcal{C}})|v-\tilde{v}| \overset{(B.0.6)}{>} \lambda_u|v-\tilde{v}|,$$

and the proof is finished.

■

Next, we recall the following known result.

Lemma B.0.13. *Let F be a continuous map of $\mathbb{R}^m$. Suppose that $F(0)=0$ and that there exist $a, b > 0$ with $|F(x)| > b$ provided that $|x| = a$. Then we have $B(b) \subset F(B(a))$ where $B(r)$ is the closed ball of $\mathbb{R}^m$ of radius r centered at the origin.*

With this, let $r_k \in K$, $B_k(\rho)$ be the closed ball of radius ρ of $U(r_k)$ and P_k, Q_k the projectors corresponding to $S(r_k)$ and $U(r_k)$ respectively. If, for every $z \in \mathbb{R}^m$ and $v \in U(r_k)$ fixed, we define

$$F_{k,z}(v) = Q_{k+1}[T(r_k+P_kz+v)-T(r_k+P_kz)], \tag{B.0.13}$$

then, we can prove the following result:

Lemma B.0.14. *Let $0 < \rho < \delta$ and $|P_kz| \leq \rho$. Then*

$$B_{k+1}(\rho\lambda_u) \subset F_{k,z}(B_k(\rho)), \tag{B.0.14}$$

with $F_{k,z}(\cdot)$ injective on $B_k(\rho)$.

Proof. By definition of $F_{k,z}$ we have that it is a continuous map and $F_{k,z}(0) = 0$. Also applying (B.0.4), if $|v| = \rho$ then

$$|F_{k,z}(v)| \geq \rho\lambda_u.$$

So, (B.0.14) is a direct consequence of Lemma B.0.13. To prove $F_{k,z}$ is injective on $B_k(\rho)$ we take $v, v' \in B_k(\rho)$. Again by (B.0.4) we conclude

$$|F_{k,z}(v) - F_{k,z}(v')| \geq \lambda_u |v - v'|. \tag{B.0.15}$$

So, $F_{k,z}(v) = F_{k,z}(v') \Rightarrow v = v'$. That is, $F_{k,z}$ is injective on $B_k(\rho)$. With this we have finished the proof.

■

With all these results, we can now prove Proposition 1.2.14:

Proof. Let $\mathcal{U}$ be a neighborhood of attractor $\mathcal{A}$, large enough, such that there exists a neighborhood $\mathcal{N}(\mathcal{A})$ with $\mathcal{N}(\mathcal{A}) \subset \overline{\mathcal{N}(\mathcal{A})} \subset \mathcal{U}$. Since T is a Morse-Smale map, we have by Lemma A.0.6 and Lemma A.0.7 that there exists a family of linear subspaces of $\mathbb{R}^m$, $(\{S(q)\}_{q\in\mathcal{U}}, \{U(q)\}_{q\in\mathcal{U}})$, which satisfies the properties described in Lemma A.0.7.

By Lemma B.0.14, the continuous map $F_{k,z}$ defined in (B.0.13), is injective on $B_k(\rho)$ and $B_{k+1}(\rho\lambda_u) \subset F_{k,z}(B_k(\rho))$. So, there exists $G_{k,z} = F_{k,z}^{-1}$, the inverse map of $F_{k,z}$ on $B_{k+1}(\rho\lambda_u)$. That is,

$$G_{k,z} : B_{k+1}(\rho\lambda_u) \longrightarrow B_k(\rho).$$

Moreover, by (B.0.15) it follows directly that

$$|G_{k,z}(w) - G_{k,z}(w')| \leq 1/\lambda_u |w - w'| \quad \text{for} \quad w, w' \in B_{k+1}(\rho\lambda_u). \tag{B.0.16}$$

Hence, we consider an arbitrary point $r \in \mathcal{N}(\mathcal{A}) \subset \overline{\mathcal{N}(\mathcal{A})}$, and we take its negative trajectory under T:

$$\mathbf{r}_- = \{r_n\}_{n\in\mathbb{Z}^-} = \{T^n(r) : n \in \mathbb{Z}^-\}.$$

Let $\{T'_n\}_{n\in\mathbb{Z}^-}$ be a family of compact maps with $T'_n : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\|T - T'_n\|_\infty = \sup_{z\in\mathbb{R}^m} |T(z) - T'_n(z)| \leq \beta \qquad \forall n \in \mathbb{Z}^-. \tag{B.0.17}$$

Our aim is to look for a point $r' \in \mathbb{R}^m$ such that its negative trajectory generated by the family $\{T'_n\}_{n\in\mathbb{Z}^-}$,

$$\mathbf{r}'_- = \{r'_n\}_{n\in\mathbb{Z}^-} = \{r'_{n+1} = T'_n(r'_n) : n \in \mathbb{Z}^-\},$$

satisfies the following inequality

$$|r_n - r'_n| \leq \alpha \|T - T'_n\|_\infty,$$

for all $n \leq 0$ for which $\mathbf{r}'_- = \{r'_n\}_{n\in\mathbb{Z}^-}$ is defined.

So, we look for a sequence $\mathbf{z}_- = \{z_n\}_{n\in\mathbb{Z}^-}$ so that, for each $n \in \mathbb{Z}^-$, $z_n \in \mathbb{R}^m$ and

$$T'_n(r_n + z_n) = r_{n+1} + z_{n+1}, \tag{B.0.18}$$

with $n \leq 0$.

Note that, if (B.0.18) holds, the sequence $\mathbf{r}'_- = \{r'_n\}_{n\in\mathbb{Z}^-} = \{r_n + z_n\}_{n\in\mathbb{Z}^-}$ is a negative trajectory of the family $\{T'_n\}_{n\in\mathbb{Z}^-}$ with

$$|r_n - r'_n| = z_n \quad n \leq 0.$$

This lead us to look for the sequence $\{z_n\}_{n\in\mathbb{Z}^-}$, as a fixed point of the following operator:

Let

$$H : Z \longrightarrow Z$$

with

$$Z = \left\{\mathbf{z}_- = \{z_n\} : n \in \mathbb{Z}^- \quad and \quad z_n \in \mathbb{R}^m\right\},$$

and

$$H(\mathbf{z}_-) = \mathbf{w}_- \quad where \quad \mathbf{w}_- = \{w_n\}, \quad n \in \mathbb{Z}^- \quad and \quad w_n = P_n w_n + Q_n w_n$$

be the operator defined by

$$P_n w_n = P_n[T'_{n-1}(r_{n-1} + z_{n-1}) - r_n] \tag{B.0.19}$$

$$Q_0 w_0 = 0 \tag{B.0.20}$$

$$\begin{aligned} Q_{n-1}w_{n-1} = \quad & G_{n-1,z_{n-1}}(Q_n[T(r_{n-1} + z_{n-1}) - T'_{n-1}(r_{n-1} + z_{n-1}) + r_n \\ & -T(r_{n-1} + P_{n-1}z_{n-1}) + z_n]), \end{aligned} \tag{B.0.21}$$

for all $n \leq 0$.

We consider the space $\mathcal{Z}$ of sequences Z

$$Z = \left\{\mathbf{z}_- = \{z_n\} : n \in \mathbb{Z}^- \quad and \quad z_n \in \mathbb{R}^m\right\},$$

with the Tikhonov product topology. Let $\mathcal{Z}_{\|T-T'_n\|_\infty}$ be the set of sequences $\mathbf{z}_- = \{z_n\}_{n\in\mathbb{Z}^-} \in \mathcal{Z}$ satisfying

$$|P_n z_n| \leq a\|T - T'_n\|_\infty \quad \textit{and} \quad |Q_n z_n| \leq b\|T - T'_n\|_\infty, \qquad \forall n \in \mathbb{Z}^-$$

with $a, b > 0$.

Since $\|T - T'_n\|_\infty \leq \beta$, see (B.0.17), if $\max\{a,b\}\beta < \delta$ we can apply estimates (B.0.1)-(B.0.4) with $K = \mathcal{N}(\mathcal{A})$. Assuming $\max\{a,b\}\beta < \delta$, we estimate the norms of the expressions which compose the definition of the operator H to check it is well defined. We start with the right side of (B.0.19),

$$D_1 := P_n[T'_{n-1}(r_{n-1} + z_{n-1}) - r_n],$$

and we write it in the following way

$$D_1 := D_{1,1} + D_{1,2} + D_{1,3},$$

with

$$\begin{aligned} D_{1,1} &:= P_n[T'_{n-1}(r_{n-1} + z_{n-1}) - T(r_{n-1} + z_{n-1})], && \text{(B.0.22)}\\ D_{1,2} &:= P_n[T(r_{n-1}+P_{n-1}z_{n-1}+Q_{n-1}z_{n-1})-T(r_{n-1}+Q_{n-1}z_{n-1})] && \text{(B.0.23)}\\ D_{1,3} &:= P_n[T(r_{n-1} + Q_{n-1}z_{n-1}) - T(r_{n-1})]. && \text{(B.0.24)} \end{aligned}$$

By property (3) of Lemma A.0.7 we have

$$|D_{1,1}| \leq \mathcal{C}\|T - T'_n\|_\infty, \qquad \forall n \in \mathbb{Z}^-.$$

By (B.0.1) we obtain

$$|D_{1,2}| \leq \lambda_s a\|T - T'_n\|_\infty, \qquad \forall n \in \mathbb{Z}^-,$$

and by (B.0.2)

$$|D_{1,3}| \leq \mu b\|T - T'_n\|_\infty \qquad \forall n \in \mathbb{Z}^-.$$

So,

$$|P_n w_n| = |D_1| \leq (\lambda_s a + \mu b + \mathcal{C})\|T - T'_n\|_\infty. \tag{B.0.25}$$

We use the same arguments as above to estimate equality (B.0.21). Now we want to study the argument of $G_{k,z}$, that is

$$D_2 := Q_n[T(r_{n-1} + z_{n-1}) - T'_{n-1}(r_{n-1} + z_{n-1}) + r_n - T(r_{n-1} + P_{n-1}z_{n-1}) + z_n],$$

and again, we write it as follows

$$D_2 := D_{2,1} + D_{2,2} + D_{2,3},$$

where

$$
\begin{aligned}
D_{2,1} &:= Q_n[T(r_{n-1}+z_{n-1}) - T'_{n-1}(r_{n-1}+z_{n-1})] && \text{(B.0.26)}\\
D_{2,2} &:= Q_n[r_n - T(r_{n-1}+P_{n-1}z_{n-1})] && \text{(B.0.27)}\\
D_{2,3} &:= Q_n[z_n]. && \text{(B.0.28)}
\end{aligned}
$$

Hence,

$$|D_{2,1}| \le \mathcal{C}\|T - T'_n\|_\infty.$$

By (B.0.3) we conclude

$$|D_{2,2}| \le \mu a\|T - T'_n\|_\infty \qquad \forall n \in \mathbb{Z}^-,$$

and

$$|D_{2,3}| \le b\|T - T'_n\|_\infty \qquad \forall n \in \mathbb{Z}^-.$$

Then,

$$|D_2| \le (\mu a + b + \mathcal{C})\|T - T'_n\|_\infty.$$

By Lemma B.0.14, if

$$(\mu a + b + \mathcal{C})\|T - T'_n\|_\infty \le b\|T - T'_n\|_\infty \lambda_u, \tag{B.0.29}$$

then the right side of (B.0.21) is defined.

Moreover, (B.0.16) implies that the right side of (B.0.21), denoted as D'_2, is estimated by

$$|Q_{n-1}w_{n-1}| = |D'_2| \le \frac{1}{\lambda_u}((\mu a + b + \mathcal{C})\|T - T'_n\|_\infty). \tag{B.0.30}$$

Let a and b be positive real numbers such that (B.0.25) y (B.0.30) are less or equal to $a\|T - T'_n\|_\infty$ and $b\|T - T'_n\|_\infty$ respectively, that is

$$(\lambda_s a + \mu b + \mathcal{C})\|T - T'_n\|_\infty \le a\|T - T'_n\|_\infty \tag{B.0.31}$$

and

$$\frac{1}{\lambda_u}((\mu a + b + \mathcal{C})\|T - T'_n\|_\infty) \le b\|T - T'_n\|_\infty. \tag{B.0.32}$$

We obtain from (B.0.31) and (B.0.32)

$$a \le \frac{\mathcal{C}(\lambda_u - 1 + \mu)}{(1-\lambda_s)(\lambda_u - 1) - \mu^2} \quad \textit{and} \quad b \le \frac{\mathcal{C}(1 - \lambda_s + \mu)}{(1-\lambda_s)(\lambda_u - 1) - \mu^2}.$$

For this reason, we take a and b as follows

$$a = \frac{\mathcal{C}(\lambda_u - 1 + \mu)}{(1-\lambda_s)(\lambda_u - 1) - \mu^2} \quad \textit{and} \quad b = \frac{\mathcal{C}(1 - \lambda_s + \mu)}{(1-\lambda_s)(\lambda_u - 1) - \mu^2}. \tag{B.0.33}$$

With this (B.0.29) holds.

Moreover, the set $\mathcal{Z}_{\|T-T'_n\|_\infty}$ is a convex and closed subset of $\mathcal{Z}$. On the other way, we have that if a and b are defined as in (B.0.33), then (B.0.25) and (B.0.30) imply that the operator H is defined on $\mathcal{Z}_{\|T-T'_n\|_\infty}$ and it maps $\mathcal{Z}_{\|T-T'_n\|_\infty}$ onto itself. Therefore, since H is a continuous operator respect to the considered topology, to apply the Schauder-Tikhonov fixed point Theorem to our operator H, we just need to prove that the image of $\mathcal{Z}_{\|T-T'_n\|_\infty}$ under the operator H is a compact subset.

With this purpose we fix $n \in \mathbb{Z}^-$. We denote

$$H_n(Z) = w_n,$$

with $Z \in \mathcal{Z}_{\|T-T'_n\|_\infty}$, $H(Z) = \{\mathbf{w}_- = \{w_n\} : n \in \mathbb{Z}^-\}$, and

$$h_n = \{H_n(Z) : Z \in \mathcal{Z}_{\|T-T'_n\|_\infty}\}.$$

We want to show h_n is precompact to conclude $H(\mathcal{Z}_{\|T-T'_n\|_\infty})$ is a compact set by the Tikhonov Theorem. With this objective we consider an arbitrary sequence $w_n^m \in h_n$ with $Z_m \in \mathcal{Z}_{\|T-T'_n\|_\infty}$ the corresponding sequence such that,

$$Z_m = \{z_n^m : n \in \mathbb{Z}^-\}, \qquad H_n(Z_m) = w_n^m.$$

Since,

$$w_n = P_n w_n + Q_n w_n,$$

and applying (B.0.19)-(B.0.21), we argue as follows. For each $n \in \mathbb{Z}^-$, we have T'_n is compact, then the points

$$v_m^1 = T'_{n-1}(r_{n-1} + z_{n-1}^m) - r_n$$

belong to a precompact subset of $\mathbb{R}^m$. And the points

$$v_m^2 = G_{n,z_n^m}(Q_{n+1}[T(r_n + z_n^m) - T'_n(r_n + z_n^m) + r_{n+1} - T(r_n + P_n z_n^m) + z_{n+1}^m])$$

belong to a bounded subset of the finite dimensional space $U(r_n)$. We remember

$$G_{n,z_n^m} : U_{n+1} \longrightarrow U_n.$$

In this way, there exist a subsequence m_l and points v^1, v^2 such that

$$v_{m_l}^i \longrightarrow v^i \quad \text{as} \quad m_l \to \infty,$$

for $i = 1, 2$.

We define w as

$$P_n w = P_n v^1 \quad Q_n w = Q_n v^2.$$

Since every map described in (B.0.19) - (B.0.21) is continuous, then

$$w_{m_l} \to w \quad \text{as} \quad m_l \to \infty.$$

Hence, h_n is precompact.

Then, by the Schauder-Tikhonov fixed point Theorem, there exists, at least, a fixed point of H. Let

$$Z = \{z_n : n \in \mathbb{Z}^-\}$$

be a fixed point of H in $\mathcal{Z}_{\|T-T'_n\|_\infty}$. For $n \in \mathbb{Z}^-$ let us show that $z_n + r_n$ is a negative trajectory of the family $\{T'_n\}_{n\in\mathbb{Z}^-}$, that is,

$$T'_{n-1}(r_{n-1} + z_{n-1}) = r_n + z_n, \qquad n \in \mathbb{Z}^-.$$

On one way,

$$P_n z_n = P_n[T'{}_{n-1}(r_{n-1} + z_{n-1}) - r_n],$$

that is,

$$P_n[z_n + r_n] = P_n[T'_{n-1}(r_{n-1} + z_{n-1})]. \tag{B.0.34}$$

On the other way, if we apply $F_{n-1,z_{n-1}}$ to (B.0.21), we obtain

$$Q_n[T(r_{n-1} + z_{n-1}) - T(r_{n-1} + P_{n-1}z_{n-1})] = Q[T(r_{n-1} + z_{n-1}) -$$

$$T'_{n-1}(r_{n-1} + z_{n-1}) + r_n - T(r_{n-1} + P_{n-1}z_{n-1}) + z_n],$$

that is,

$$Q_n[r_n + z_n] = Q_n[T'_{n-1}(r_{n-1} + z_{n-1})]. \tag{B.0.35}$$

If we put together (B.0.34) and (B.0.35) we have the desired result,

$$T'_{n-1}(r_{n-1} + z_{n-1}) = r_n + z_n = r'_n.$$

We denote

$$\theta_1 = \sup_{k\le 0} |P_k z_k| \qquad \text{and} \qquad \theta_2 = \sup_{k\le 0} |Q_k z_k|.$$

Since we have seen before, if a and b are defined as in (B.0.33) then

$$|P_n z_n| \le a\|T - T'_n\|_\infty \quad \text{and} \quad |Q_n z_n| \le b\|T - T'_n\|_\infty.$$

Therefore we have

$$\theta_1 = a\|T - T'_n\|_\infty \quad \text{and} \quad \theta_2 = b\|T - T'_n\|_\infty.$$

With this we have obtained

$$\sup_{k\le 0} |z_k| \le \theta = (a + b)\|T - T'_n\|_\infty,$$

with

$$a + b = \mathcal{C}\frac{2\mu + \lambda_u - \lambda_s}{(1 - \lambda_s)(\lambda_u - 1) - \mu^2}.$$

That is, we have found a negative trajectory of the family $\{T'_n\}_{n\in\mathbb{Z}^-}$,

$$T'_{n-1}(r'_{n-1}) = r'_n, \qquad n \in \mathbb{Z}^-,$$

with $\|T - T'_n\|_\infty \leq \beta$ such that,

$$|r_n - r'_n| \leq \alpha \|T - T'_n\|_\infty,$$

with $\mathbf{r}_- = \{r_n\}_{n\in\mathbb{Z}^-}$ a negative trajectory under T of an arbitrary point $r \in \mathcal{N}(\mathcal{A})$ and

$$a\beta \leq \delta \qquad \text{and} \qquad b\beta \leq \delta,$$

that is,

$$\beta = \min\{\frac{1}{a}, \frac{1}{b}\}\delta = \frac{(1-\lambda_s)(\lambda_u - 1) - 2\mu}{\max\{(\mu + \lambda_u - 1), (1 - \lambda_s + \mu)\}}\mathcal{C}^{-1}\delta.$$

This implies that T has the *Nonautonomous Inverse Shadowing* property on $\mathcal{N}(\mathcal{A})$ with parameters

$$\alpha = \mathcal{C}\frac{2\mu + \lambda_u - \lambda_s}{(1-\lambda_s)(\lambda_u - 1) - \mu^2}$$

and

$$\beta = \frac{(1-\lambda_s)(\lambda_u - 1) - \mu^2}{\max\{(\mu + \lambda_u - 1), (1 - \lambda_s + \mu)\}}\mathcal{C}^{-1}\delta$$

as we wanted to prove.

$\blacksquare$

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